A surrealistic tale of electricity

Hans U. Fuchs
Winterthur Polytechnic (School of Engineering), 8401 Winterthur, Switzerland

(Received 15 April 1985; accepted for publication 14 October 1985)

Using thermodynamics as an analogy, an "electricity machine" is constructed and analyzed. This machine runs through a "Carnot" cycle. The analysis leads to the construction of a "new" quantity called "reduced electricity" (charge). The example shows what life in electrodynamics would be without the substancelike quantity which we call electrical charge. (This state of affairs is accepted in thermodynamics.) Imagine the development of the theory of electricity had fallen into the hands of the thermodynamicists. As fairy and other tales go, they have an air of surrealism surrounding them.

I. THE EARLY YEARS

In a quite real world, some almost real physicists studied electricity. They had already learned how to measure and how to deal with an electric quantity which they called voltage. They gave it the symbol U. However, they were not quite sure at all what to think about electricity. Were electricity and voltage the same? Or not? What was electricity after all?

Engineers started to build electricity machines, as they were called. The machines were used for different purposes, to do different types of work. Their efficiency was quite low, so a young Frenchman (S. Carnot) addressed himself to some theoretical questions. He studied the machines and concluded that the efficiency of an ideal machine could approach 100%. In his mind he had a pretty clear picture of what electricity was like; this helped him to deduce his results. For him, electricity could be compared to water which, when falling from a higher to a lower level, would do work. In electricity, the level would be the voltage. He derived his results without knowing the law of conservation of energy.

Not everybody was happy with this interpretation of electricity. Count Rumford, an American by the way, noticed that electricity could be produced using a dynamo attached to his ten-speed bicycle (he was particularly fond of his microprocessor controlled gear shift). You could produce electricity as long as you rode the bike. It seemed difficult to think of electricity as a "substance," a "fluidum," which could be created continuously.

Soon afterward, the law of conservation of energy was discovered. For the physicists, it became obvious that electricity had to be energy, an energy form. You could not "create" electricity with a dynamo. Electricity simply was converted work. The dynamo was a machine that converted work into electricity. It did not take long, and the *First Law of Electricity* was formulated. The (internal) energy (E_i) of a system could be changed in two ways: by adding (or removing) electricity (symbol K) and by doing work (W):

$$dE_i = dK + dW. \quad (1)$$

It became obvious to everybody involved that Carnot's idea concerning the nature of electricity was cute but wrong.

II. THE GOLDEN YEARS

A German physicist (R. Clausius) started to analyze electricity machines from this new point of view. Especially, he investigated the capacitor with variable separation as a concrete example of a machine that could convert electricity into work. A couple of things were known about this physical system at the time (the following holds approximately for large capacitor plates with small separation). One important result related the force (F) acting between the capacitor plates to the separation x and the voltage U. This was called the *equation of state* of the ideal parallel plate capacitor:

$$F x^2 = \tfrac{1}{2}\epsilon_0 A\, U^2. \quad (2)$$

(See Fig. 1.) Here, A was the area of the capacitor plates and ϵ_0 was a fundamental constant. A second result was this: when the capacitor was *not* hooked up to a battery, the force F between the plates did not depend on the separation x (if the separation was kept small). Separation and voltage changed in such a way as to leave F unchanged. A third point could be deduced from the observation just mentioned. The energy of a parallel plate capacitor had to be

$$E_i = F x = \tfrac{1}{2}\epsilon_0 A (U^2/x). \quad (3)$$

Clausius calculated a particular closed cycle of this machine. He composed the cycle of two isovoltaic steps (constant voltage) and two anelectric steps (for an anelectric step $\Delta K = 0$!; see Fig. 2). In an F-x diagram, the four steps can easily be visualized (Fig. 2). Steps 1 and 3 are the anelectric ones, step 2 is isovoltaic at U_2, step 4 is isovoltaic at U_1. The work given off by this machine could be calculated by the shaded area in the F-x diagram.

The processes this machine could undergo were calcu-

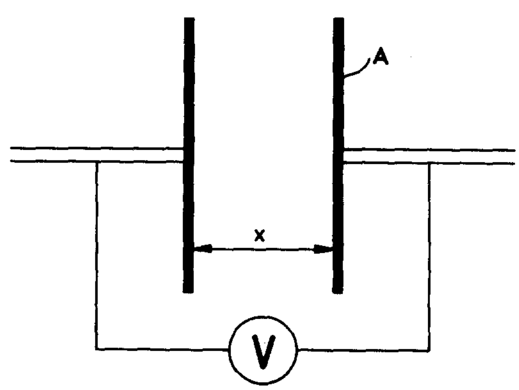

Fig. 1. A parallel plate capacitor with variable separation x.

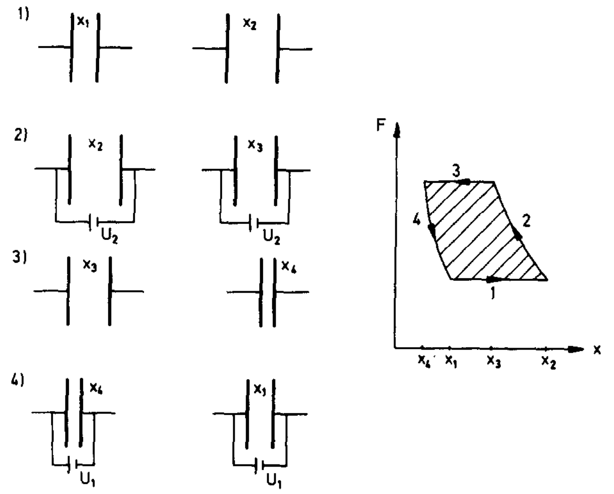

Fig. 2. An "electricity machine." A parallel plate capacitor undergoes a four-step closed cycle. In each step, the separation of the two plates is changed. The cycle is represented again in the *F-x* (force-separation) diagram. The shaded area enclosed by the four steps represents the work done by this machine on the surroundings in one cycle.

lated using the First Law. From (3) we have

$$dE = \frac{\partial E}{\partial U} dU + \frac{\partial E}{\partial x} dx = \epsilon_0 A \frac{U}{x} dU - \frac{1}{2} \epsilon_0 A \frac{U^2}{x^2} dx.$$

With $dW = Fdx$, dK turns out to be

$$dK = \epsilon_0 A \frac{U}{x} dU - \frac{1}{2} \epsilon_0 A \frac{U^2}{x^2} dx - \frac{1}{2} \epsilon_0 A \frac{U^2}{x^2} dx$$

or

$$dK = \epsilon_0 A \frac{U}{x} dU - \epsilon_0 A \frac{U^2}{x^2} dx. \tag{4}$$

The two processes, anelectric and isovoltaic, can now be computed. For anelectric processes we find:

$$dK = 0 = \epsilon_0 A \left(\frac{U}{x} dU - \frac{U^2}{x^2} dx \right)$$

or

$$\frac{dU}{U} = \frac{dx}{x}$$

or

$$U \sim x. \tag{5}$$

For isovoltaic processes, the following result holds (since $dU = 0$):

$$dK = -\epsilon_0 A \frac{U^2}{x^2} dx. \tag{6}$$

In our particular case this means:

$$\Delta K_2 = -\epsilon_0 A\, U_2^2 \int_{x_2}^{x_3} \frac{dx}{x^2} = \epsilon_0 A\, U_2^2 \left(\frac{1}{x_3} - \frac{1}{x_2} \right), \tag{7}$$

$$\Delta K_1 = -\epsilon_0 A\, U_1^2 \int_{x_4}^{x_1} \frac{dx}{x^2} = \epsilon_0 A\, U_1^2 \left(\frac{1}{x_1} - \frac{1}{x_4} \right). \tag{8}$$

Clausius completed the calculations by noting that x_1 and x_2 and x_3 and x_4 lie on different "anelectrics." With (5) he concluded that

$$x_1/x_2 = U_1/U_2, \quad x_4/x_3 = U_1/U_2. \tag{9}$$

Now, he had the decisive insight. Using (9), he concluded from (7) and (8) that the "*reduced electricities*" $\Delta K/U$ were the same (except for the sign) for both isovoltaic steps in what became known as the *Carnot cycle* of electricity machines by physicists:

$$\frac{\Delta K_2}{U_2} = \epsilon_0 A \left(\frac{U_2}{x_3} - \frac{U_2}{(U_2/U_1)\, x_1} \right) = \epsilon_0 A \left(\frac{U_2}{x_3} - \frac{U_1}{x_1} \right),$$

$$\frac{\Delta K_1}{U_1} = \epsilon_0 A \left(\frac{U_1}{x_1} - \frac{U_1}{(U_1/U_2)\, x_3} \right) = \epsilon_0 A \left(\frac{U_1}{x_1} - \frac{U_2}{x_3} \right),$$

or, equivalently

$$\frac{\Delta K_1}{U_1} + \frac{\Delta K_2}{U_2} = 0. \tag{10}$$

This result could be extended to general cycles of an electricity machine; it could then be written in the elegant form

$$\oint \frac{dK}{U} = 0. \tag{11}$$

Even though electricity (dK) is not a state variable, Clausius concluded that the reduced electricity dK/U was a *state variable*. As such it had to play an important role in the formal development of electrodynamics. Because of its importance, Clausius gave dK/U its own name. He searched long in some old languages and finally called the new state variable *ladung*[1] and gave it the symbol Q. Today, nobody really knows why this word was chosen. The formal definition of *ladung* is

$$\Delta Q = \int \frac{dK}{U}. \tag{12}$$

Even though *ladung* was only a mathematical construction, it soon became evident that something important had been discovered. It turned out that *ladung* was a conserved quantity in all natural processes. One soon formulated the Second Law of Electricity:

In a closed system, the total *ladung* remains constant in all processes.

Consequently, electrodynamics was formally developed around the theory proposed by Clausius. One nice little result concerns the efficiency of the Carnot cycle. Using the developments that lead to Eq. (10), it can be shown that the efficiency is given by

$$\eta_C = (U_2 - U_1)/U_2. \tag{13}$$

It is quite simple (in theory at least) to make the voltage U_1 small compared to U_2. This lets the efficiency approach 100%.

III. EPILOGUE

Today, there exist many teachers of physics who place the concept of *ladung* at the center of their teaching of electrodynamics. They maintain that *ladung* is a substancelike quantity which would make it easy to visualize and deal with. It has even been proposed to call *ladung* "charge" or "electricity," just as Carnot had done 150 years earlier.

However, if I may venture an opinion, I do not see any need for replacing the beautiful formalism developed by Clausius by something else. Just because the concept of "electric charge" might be so plausible and graphic does not mean that we should teach physics that way.