Natur, Technik, Systeme 2: Semesterend-Prüfung, Juni 2013

Zweites Semester Wirtschaftsingenieurwesen, ZHAW, WI12
dumo, fusa, furu, hsng, lucs

General Remarks

- Duration of the exam: 150 minutes.
- Answers must be explained and must be documented.
- Allowed tools: Books and personally written summary, Lecture Notes. Calculators and writing materials.
- Please solve each problem on a separate sheet. The problems must be handed in individually!
- Write your name, date, exam, and number of problem on every sheet.
- Hand in the problem statements with your solutions. Write your name on the problem statements!
- Points:

  Problem 1: 13

  Problem 2: 12

  Problem 3: 15

1. Investigation of the bahavior of dynamical systems.

   (a) Express the differential equation

   $$\ddot{x} - \dot{x} + c = 0 \tag{1}$$

   in terms of two differential equations of 1st order. [2 P.]

   (b) [For WI12a,c,d,t] The differential equation in Problem a is supposed to be solved with Matlab. The initial conditions are $x(0) = 1$ and $\dot{x}(0) = 2$. The call of `ode45` will look like this:

   ```
   [t,x] = ode45('DGLs',ts,x0);
   ```

   Write the function `DGLs` and the vector `x0` in syntactically correct forms. [2 P.]

   [For WI12b] You would like to solve the differential equation in Problem a with the help of Berkeley Madonna. The initial conditions are $x(0) = 1$ and $\dot{x}(0) = 2$. Draw the flow chart for this problem. The contents of all elements must be shown. [2 P.]

   (c) Consider the differential equation

   $$\dot{x} = x(x+1)(x-1) \tag{2}$$

   Assume an initial value of $x_0 = 0.6$ and compute the numerical solution of $x(t)$ taking steps of 0.1 up to $t = 1$. Use the *explicity* Euler method. [3 P.]

   (d) Sketch the 1D vector field of Equ.(2) for für $-3/2 \leq x \leq 3/2$. Your sketch is supposed to show the sign and the relative intensity of the dynamic behavior. Moreover, the equilibrium points (fix points) of the system must be visible. Hint: Make the analysis of the vector field for about 10 different values of x. [3 P.]

   (e) Explain the behavior of the system for $t \to \infty$ for the following initial values [3 P.]:

   - $x_0 = 3/2$
   - $x_0 = 1$
   - $x_0 = 1/2$
   - $x_0 = 0$
   - $x_0 = -0.4$
   - $x_0 = -1$
   - $x_0 = -1.01$

2. *General situation:* A heat pump for the hot water supply of a house is to be designed. The entropy is taken from the layers of soil in the garden. We want to know how large the collecting area in the garden has to be for the coefficient of performance (cop) not to go below 4 for a well defined condition.

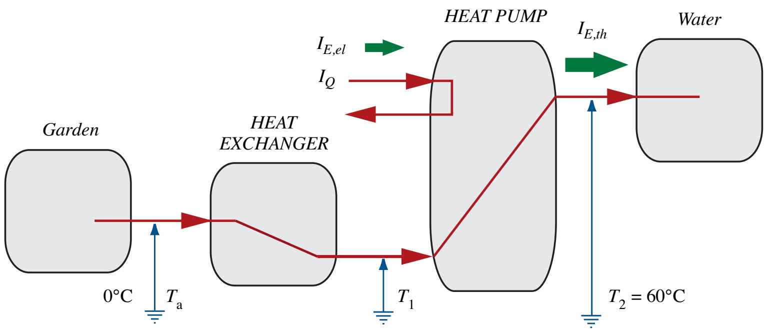
   Special situation: The heat pump heats water in a 300 liter boiler. At $t = 0$, the temperature of the water equals 60°C; we want it to rise at a rate of 10°C per hour. The temperature of the soil is equal to 0°C.

   Model: Heat transfer at the hot end of the heat pump is close to perfect (i.e., we do not need a heat exchanger there in our model). At the cold end, however, the layers of soil and the pipes for the working fluid of the heat pump form a heat exchanger having a heat transfer coefficient of $h = 1.0$ W/(K·m^2). The heat pump itself works reversibly (endoreversible model).

   Coefficient of performance: The coefficient of performance (cop) is the ratio (see figure)

$$cop = \frac{I_{E,th}}{I_{E,el}} \tag{3}$$

   Data: Specific heat of water: $c = 4200$ J/(K·kg).



   Figur: *Process diagram of the proposed model having a single heat exchanger. Entropy currents and temperatures are shown. Only the energy currents needed for defining the coefficient of performance have been drawn in the diagram.*

   (a) Determine the entropy current necessary for heating the water in the boiler according to specifications (at $t = 0$). [3 P]

   (b) Determine the electric power necessary for the cop to be exactly equal to 4. [3 P]

   (c) How low is the temperature T_1 at the exit of the heat exchanger, i.e., at the entrance to the heat pump? [3 P]

   (d) How large does the collecting area in the garden have to be? [3 P]

3. *General situation:* A certain amount of radioactive iodine (I-131) has been taken from a reactor and is placed in a water pool. You must formulate and investigate the initial value problem for the amount of iodine and the temperature of the water in the pool.

Concrete questions:

(a) The decay rate (reaction rate) of the radioactive material is proportional to the instantaneous amount $n(t)$. Formulate the initial value problem for $n(t)$ using the time constant and then show that the solution of this initial value problem must be:

$$n(t) = n_0 \exp(-t/\tau) \tag{4}$$

where τ is the time constant and $n_0 = n(0)$. [4 P.]

(b) Show that the rate at which energy is released (the power $\mathscr{P}$) as a consequence of the decay is given by

$$\mathscr{P}(t) = \frac{\bar{e}}{\tau} n_0 \exp(-t/\tau) \tag{5}$$

where $\bar{e}$ is the energy released in the decay of 1 mol of I-131. [2 P.]

(c) The iodine is in a cubic water pool having edges of length L (volume $= L^3$). The initial temperature of the water is equal to the ambient temperature. The energy released by the iodine goes directly into the water and is dissipated there. Heat loss occurs only through the surface of the pool ($A = L^2$) und obeyes a linear relation for the energy current with an energy conductance of $G_E = Ah$. Show that the initial value problem for the temperature of the water is given by

$$\frac{dT}{dt} = \frac{\bar{e}}{\tau \rho L^3 c} n_0 \exp(-t/\tau) - \frac{h}{\rho L c}(T - T_a) \quad , \quad T(0) = T_a \tag{6}$$

where c is the specific heat of water, ρ is its density, and T_a is the ambient temperature. [4 P.]

(d) Explain what type of differential equation Equ.(6) is (order, linearity, homogeneity; is it autonomous?). [2 P.]

(e) Calculate $T(t)$ for the case of constant decay rate $\mathit{\Pi}_n$ of iodine (this is not realistic but makes the problem mathematically simple). [3 P.]

(f) [Additional problem] The initial value problem Equ.(6) has the generic form

$$\frac{dy}{dt} = a \exp(-kt) + b - py \quad , \quad y(0) = y_0 \tag{7}$$

Determine the coefficients a, b, p and k and the initial value y_0 in terms of the physical quantities used so far. [2 P.]

(g) [Additional problem] The differential equation(7) has the solution

$$y(t) = \frac{ap \exp(-kt)}{p^2 - pk} + \frac{bp}{p^2 - pk} - \frac{bk}{p^2 - pk} + \alpha \exp(-pt) \tag{8}$$

Determine the factor α and rewrite the solution with this new result (using factors a, b, p and k, and y_0). [3 P.]

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Solutions

1. Investigation of the bahavior of dynamical systems.

   (a) Express the differential equation

$$\ddot{x} - \dot{x} + c = 0 \tag{1}$$

   in terms of two differential equations of 1st order. [2 P.]

   Solution:

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = x_2 - c$$

   (b) The differential equation in Problem a is supposed to be solved with Matlab. The initial conditions are $x(0) = 1$ and $\dot{x}(0) = 2$. The call of `ode45` will look like this:

```
[t,x] = ode45('DGLs',ts,x0);
```

   Write the function `DGLs` and the vector `x0` in syntactically correct forms. [2 P.]

   Solution:

```
x0 = [1 2]
function xdot = DGLs(t,x)
global c
xdot(1) = x(2);
xdot(2) = x(2) - c;
xdot = xdot';
```

   (c) Consider the differential equation

$$\dot{x} = x(x+1)(x-1) \tag{2}$$

   Assume an initial value of $x_0 = 0.6$ and compute the numerical solution of $x(t)$ taking steps of 0.1 up to $t = 1$. Use the *explicity* Euler method. [3 P.]

Solution:

```
xi+1 = xi + xi*(xi+1)*(xi-1)*△t
```

t	$x(t)$	t	$x(t)$
0	0.6	0.6	0.3780
0.1	0.5616	0.7	0.3456
0.2	0.5232	0.8	0.3152
0.3	0.4852	0.9	0.2868
0.4	0.4481	1.0	0.2605
0.5	0.4122		

(d) Sketch the 1D vector field of Equ.(2) for für $-3/2 \leq x \leq 3/2$. Your sketch is supposed to show the sign and the relative intensity of the dynamic behavior. Moreover, the equilibrium points (fix points) of the system must be visible. Hint: Make the analysis of the vector field for about 10 different values of x. [3 P.]

Solution:

(e) Explain the behavior of the system for $t \to \infty$ for the following initial values [3 P.]:

Solution:

- $x_0 = 3/2 \to x_\infty = +\infty$
- $x_0 = 1 \to x_\infty = 1$
- $x_0 = 1/2 \to x_\infty = 0$
- $x_0 = 0 \to x_\infty = 0$
- $x_0 = -0.4 \to x_\infty = 0$
- $x_0 = -1 \to x_\infty = -1$
- $x_0 = -1.01 \to x_\infty = -\infty$

2. *General situation:* A heat pump for the hot water supply of a house is to be designed. The entropy is taken from the layers of soil in the garden. We want to know how large the collecting area in the garden has to be for the coefficient of performance (*cop*) not to go below 4 for a well defined condition.

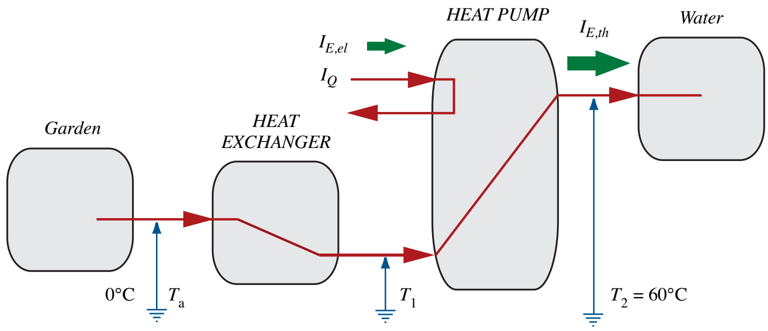
   Special situation: The heat pump heats water in a 300 liter boiler. At $t = 0$, the temperature of the water equals 60°C; we want it to rise at a rate of 10°C per hour. The temperature of the soil is equal to 0°C.

   Model: Heat transfer at the hot end of the heat pump is close to perfect (i.e., we do not need a heat exchanger there in our model). At the cold end, however, the layers of soil and the pipes for the working fluid of the heat pump form a heat exchanger having a heat transfer coefficient of $h = 1.0$ W/(K·m^2). The heat pump itself works reversibly (endoreversible model).

Coefficient of performance: The coefficient of performance (cop) is the ratio (see figure)

$$cop = \frac{I_{E,th}}{I_{E,el}} \tag{3}$$

Data: Specific heat of water: $c = 4200$ J/(K·kg).



Figur: *Process diagram of the proposed model having a single heat exchanger. Entropy currents and temperatures are shown. Only the energy currents needed for defining the coefficient of performance have been drawn in the diagram.*

(a) Determine the entropy current necessary for heating the water in the boiler according to specifications (at $t = 0$). [3 P]

Solution:

$$\dot{S}_{water} = I_S$$
$$\dot{S}_{water} = K\,\dot{T}$$
$$K = \frac{C}{T} = \frac{mc}{T}$$

$\Rightarrow$

$$I_S = \frac{mc}{T}\,\dot{T} = \frac{300 \cdot 4200}{333}\,\frac{10}{3600}\,\mathrm{W/K} = 10.5\,\mathrm{W/K}$$

(b) Determine the electric power necessary for the cop to be exactly equal to 4. [3 P]

Solution:

$$cop = \frac{I_{E,th}}{I_{E,el}} = \frac{I_{E,th}}{\mathscr{P}_{el}}$$
$$I_{E,th} = T\,I_S$$

$\Rightarrow$

$$\mathscr{P}_{el} = \frac{I_{E,th}}{cop} = \frac{T\,I_S}{cop} = \frac{333 \cdot 10.5}{cop}\,\mathrm{W} = 874\,\mathrm{W}$$

(c) How low is the temperature T_1 at the exit of the heat exchanger, i.e., at the entrance to the heat pump? [3 P]

Solution:

$$\mathscr{P}_{el} = (T_2 - T_1)\, I_S$$

$\Rightarrow$

$$T_1 = T_2 - \frac{\mathscr{P}_{el}}{I_S} = 333\mathrm{K} - \frac{874}{10.5}\mathrm{K} = 250\mathrm{K}$$

(d) How large does the collecting area in the garden have to be? [3 P]

Solution:

$$I_{E,th,cold-end} = T_1 I_S$$
$$I_{E,th,cold-end} = Ah\,(T_a - T_1)$$

$\Rightarrow$

$$A = \frac{T_1 I_S}{h\,(T_a - T_1)} = \frac{250{\cdot}10.5}{1.0{\cdot}(273 - 250)}\mathrm{m}^2 = 114\,\mathrm{m}^2$$

3. *General situation:* A certain amount of radioactive iodine (I-131) has been taken from a reactor and is placed in a water pool. You must formulate and investigate the initial value problem for the amount of iodine and the temperature of the water in the pool.

   Concrete questions:

   (a) The decay rate (reaction rate) of the radioactive material is proportional to the instantaneous amount $n(t)$. Formulate the initial value problem for $n(t)$ using the time constant and then show that the solution of this initial value problem must be:

$$n(t) = n_0 \exp(-t/\tau) \tag{4}$$

   where τ is the time constant and $n_0 = n(0)$. [4 P.]

   Solution:

$$\frac{dn}{dt} = -\Pi_n \quad , \quad n(0) = n_0$$
$$\Pi_n = k\,n = \frac{1}{\tau}n$$

$\Rightarrow$

$$\frac{dn}{dt} = -\frac{1}{\tau}n \quad , \quad n(0) = n_0$$

   Separation of variables:

$$\int_{n_0}^{n} \frac{dn}{n} = \int_{t=0}^{t} -\frac{dt}{\tau}$$

$$\ln\left(\frac{n}{n_0}\right) = -\frac{t}{\tau} \quad \Rightarrow \quad n(t) = n_0 \exp\left(-\frac{t}{\tau}\right)$$

(b) Show that the rate at which energy is released (the power $\mathscr{P}$) as a consequence of the decay is given by

$$\mathscr{P}(t) = \frac{\bar{e}}{\tau} n_0 \exp(-t/\tau) \tag{5}$$

where $\bar{e}$ is the energy released in the decay of 1 mol of I-131. [2 P.]

Solution:

$$\mathscr{P}(t) = \bar{e}\, \Pi_n$$

$$\Pi_n = -\frac{dn}{dt} = -\frac{d}{dt}\left(n_0 \exp(-{}^t/_\tau)\right) = \frac{1}{\tau} n_0 \exp\left(-\frac{t}{\tau}\right)$$

$\Rightarrow$

$$\mathscr{P}(t) = \frac{\bar{e}}{\tau} n_0 \exp(-t/\tau)$$

(c) The iodine is in a cubic water pool having edges of length L (volume = L^3). The initial temperature of the water is equal to the ambient temperature. The energy released by the iodine goes directly into the water and is dissipated there. Heat loss occurs only through the surface of the pool ($A = L^2$) und obeyes a linear relation for the energy current with an energy conductance of $G_E = Ah$. Show that the initial value problem for the temperature of the water is given by

$$\frac{dT}{dt} = \frac{\bar{e}}{\tau \rho L^3 c} n_0 \exp(-t/\tau) - \frac{h}{\rho L c}(T - T_a) \quad , \quad T(0) = T_a \tag{6}$$

where c is the specific heat of water, ρ is its density, and T_a is the ambient temperature. [4 P.]

Solution:
Law of balance of energy for water in pool:

$$\frac{dE}{dt} = \mathscr{P}(t) - I_{E,th,loss}(t)$$

$$I_{E,th,loss} = G_E\left(T(t) - T_a\right) \quad , \quad G_E = L^2 h$$

$$\frac{dE}{dt} = C\frac{dT}{dt} \quad , \quad C = mc = \rho L^3 c$$

$\Rightarrow$

$$\rho L^3 c \frac{dT}{dt} = \frac{\bar{e}}{\tau} n_0 \exp(-t/\tau) - L^2 h\left(T(t) - T_a\right)$$

or

$$\frac{dT}{dt} = \frac{\bar{e}}{\tau \rho L^3 c} n_0 \exp(-t/\tau) - \frac{L^2 h}{\rho L^3 c}\left(T(t) - T_a\right) \quad , \quad T(0) = T_a$$

(d) Explain what type of differential equation Equ.(6) is (order, linearity, homogeneity; is it autonomous?). [2 P.]

Solution:
1st order DE in $T(t)$. Linear. Inhomogeneous. Non-autonomous.

(e) Calculate $T(t)$ for the case of constant decay rate $\varPi_n$ of iodine (this is not realistic but makes the problem mathematically simple). [3 P.]

Solution:

$$\frac{dT}{dt} = a - b\left(T(t) - T_a\right)$$

$$a = \frac{\bar{e}\,\varPi_n}{\rho\, L^3 c} \quad , \quad b = \frac{h}{\rho\, L\, c}$$

Separation of variables:

$$T(t) = \frac{a}{b}\left(1 - \exp\left(-bt\right)\right) + T_a$$

or

$$T(t) = \frac{\bar{e}\,\varPi_n}{L^2 h}\left(1 - \exp\left(-\frac{h}{\rho\, L\, c}t\right)\right) + T_a$$

(f) [Additional problem] The initial value problem Equ.(6) has the generic form

$$\frac{dy}{dt} = a\,\exp\left(-kt\right) + b - py \quad , \quad y(0) = y_0 \tag{7}$$

Determine the coefficients a, b, p and k and the initial value y_0 in terms of the physical quantities used so far. [2 P.]

Solution:

$$\frac{dT}{dt} = \frac{\bar{e}}{\tau\rho\, L^3 c} n_0 \exp\left(-t/\tau\right) - \frac{L^2 h}{\rho\, L^3 c}\left(T(t) - T_a\right) \quad , \quad T(0) = T_a$$

$$\frac{dy}{dt} = a\, exp\left(-kt\right) + b - py \quad , \quad y(0) = y_0$$

$\Rightarrow$

$$a = \frac{\bar{e}}{\tau\rho\, L^3 c} n_0 \quad , \quad b = \frac{h}{\rho\, L\, c} T_a \quad , \quad p = \frac{h}{\rho\, L\, c}$$

$$k = \frac{1}{\tau} \quad , \quad y_0 = T_a$$

(g) [Additional problem] The differential equation(7) has the solution

$$y(t) = \frac{ap\,\exp\left(-kt\right)}{p^2 - pk} + \frac{bp}{p^2 - pk} - \frac{bk}{p^2 - pk} + \alpha\,\exp\left(-pt\right) \tag{8}$$

Determine the factor α and rewrite the solution with this new result (using factors a, b, p and k, and y_0). [3 P.]

Solution:
Insert the initial condition for $t = 0$:

$$y\left(0\right) = \frac{ap}{p^2 - pk} + \frac{bp}{p^2 - pk} - \frac{bk}{p^2 - pk} + \alpha$$

$$\alpha = y_0 - \left(\frac{ap}{p^2 - pk} + \frac{bp}{p^2 - pk} - \frac{bk}{p^2 - pk}\right)$$

$\Rightarrow$

$$y\left(t\right) = \frac{ap\exp\left(-kt\right)}{p^2 - pk} + \frac{bp}{p^2 - pk} - \frac{bk}{p^2 - pk} + \left[y_0 - \left(\frac{ap}{p^2 - pk} + \frac{bp}{p^2 - pk} - \frac{bk}{p^2 - pk}\right)\right]\exp\left(-pt\right)$$