CHAPTER 1

STORAGE AND FLOW OF FLUIDS

Fluids present us with some of the most vivid and most easily studied physical processes. What we learn here will carry over into most of the other fields such as electricity, heat, chemical substances, and motion.

The image upon which models of fluid processes are based has two major aspects. One, dynamical fluid phenomena are the result of storage and flow of quantities of fluid, and two, pressure differences are the driving forces of these processes.

1.1 SOME IMPORTANT OBSERVATIONS

Equilibration of fluid levels. If we fill two tanks to different levels with oil or water, and if we connect them through a hose at the bottom (Fig. 1.1), we observe that the level which is higher decreases, while the one that is lower increases. This happens as long as the levels are different. The process runs fast at the beginning and then slows down. This happens no matter how big—or how wide—the tanks are, or what shape they have.

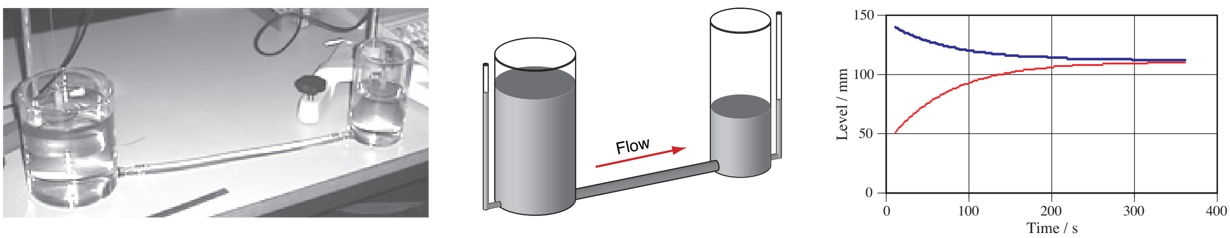

Figure 1.1: Two tanks containing rape seed oil are connected by a hose at the bottom, the liquid flows from the one having the higher fluid level to the one having a lower level (this is independent of the size of the tanks; levels equilibrate, not quantities of liquid). Right: Data for the levels of rape seed oil.

Interpretation. Liquids are stored in tanks, and they can flow from tank to tank. For the liquid to flow, level differences are needed. We say that a level difference drives the flow of liquid. When the level difference is high (as at the beginning), the flow is high. Inflow and outflow of liquids lead to changes of the quantities stored in the tanks. A strong net flow will lead to a fast change of the quantity stored.

Equilibrating pressures. If oil is filled in one of two communicating tanks, and water in the other, levels will be different after equilibration has taken place (Fig. 1.2). In another experiment, two balloons are connected by a hose having a valve (Fig. 1.2). Both balloons are filled with air, one to a high pressure, the other to a somewhat lower one. When the valve is opened, air flows from the balloon having the higher air pressure into the other one.

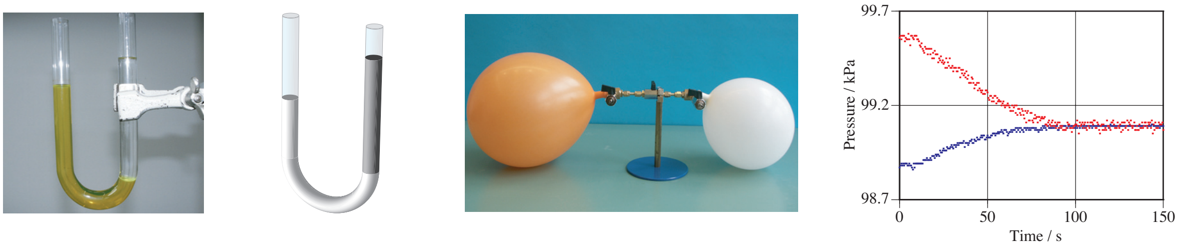

Figure 1.2: With oil on top of water in the right column of the U-pipe, the levels of the liquids are not the same after equilibration. The effect is observed as well if the two tanks have different cross sections and/or different shapes. When air is allowed to flow freely between two communicating balloons, neither volumes nor levels are the same after equilibration. As the diagram on the right shows, the pressure of the air in the balloons equilibrates. The upper curve in the diagram belongs to the smaller balloon (surprisingly!).

Interpretation. Nature does not equilibrate levels nor volumes but the pressure of fluids. This is seen when pressure gauges at the bottoms of the tanks in Fig. 1.1 are used. The same result is obtained if different liquids are filled into the tanks as in the U-pipe of Fig. 1.2, or if the pressure of air in communicating balloons is measured (Fig. 1.2, diagram on the right). Consequently, we can interpret the pressure difference in the communicating containers as the driving force for flows.

Gravity and pressure differences. If we fill water or another liquid into a tank and measure the pressure of the fluid at the bottom, we notice that this value increases as we add more and more of the fluid. Observation demonstrates that the pressure increases linearly with depth. This finding is independent of the size and shape of the container. In fact, we would measure precisely the same linear relation between pressure and depth in a lake, in a narrow tank, or even in the ocean! (See Fig. 1.3.)

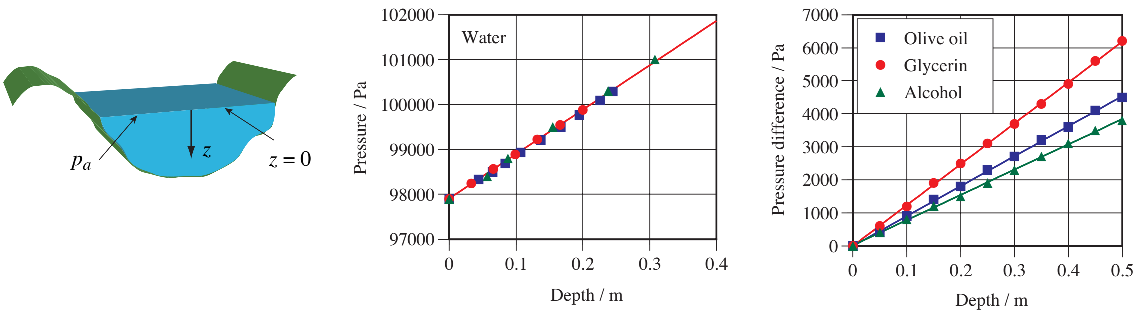

Figure 1.3: Pressure versus depth in water in tanks of different cross sections (diagram at center). The slope of the linear fit is close to 10,000 SI units. The pressure difference measured from the surface as a function of depth in different liquids (right). The linear fits have different slopes (higher density yields steeper slope).

The slope of the straight pressure–depth relation does depend upon the density of the liquid, however. In water it increases by 1 bar every 10 meters (1 bar = 10^5 Pascals). In a typical vegetable oil this would be reduced to about 0.9 bar for a 10 meter column. It would be 13.6 bar for 10 meters of mercury.

In the Earth's atmosphere, the pressure decreases upward, just as in a lake, but the pressure-height relation is not linear (Fig. 1.4). The pressure at sea level changes by about 0.001 bar every 10 meters, and it changes less quickly at higher altitudes.

Interpretation. Fluids press down upon layers below them as a result of their weight. That explains why denser fluids have a steeper pressure gradient. (A gradient measures how quickly a quantity changes in a certain direction. So it is a kind of "spatial rate of change.") Apparently, the pressure gradient in a fluid due to gravity is proportional to the density of the fluid (Fig. 1.3). In Fig. 1.4 we see that the density of the air in the atmosphere decreases with height above the ground.

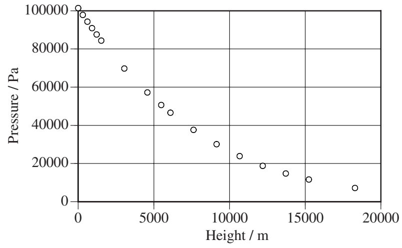

Figure 1.4: Pressure as a function of height in the atmosphere. Values correspond to the so-called standard atmosphere. In reality, conditions change with time and place.

Experience shows that the pressure-height relations in fluids at rest do not depend upon the form or size of the containers or the horizontal extend of the fluid (see Fig. 1.3, center).The pressure gradient for a fluid is the same in a narrow tank and in a wide one. We conclude that pressure has to be the same in a liquid at a given height since there is no horizontal flow.

Mercury and water barometers and pressure sensors. A barometer is a pressure sensor measuring the pressure of the air in the atmosphere. In 1643, Evangelista Torricelli first built a water barometer that needed a pipe having a height of some 10 meters. Using mercury, he could build the barometer much shorter, less than 1 meter (Fig. 1.5).

A barometer is built as follows. A liquid such as water or mercury is filled into a thin tube closed at one end. The tube is inverted and placed into a bowl containing the same liquid. In the case of mercury, the column of liquid drops to a height of about 75 cm above the level in the bowl, leaving an empty space above. Instead of a pipe stuck in a bowl, we can also use a bent pipe (Fig. 1.5, right) with the short leg open to the ambient air.

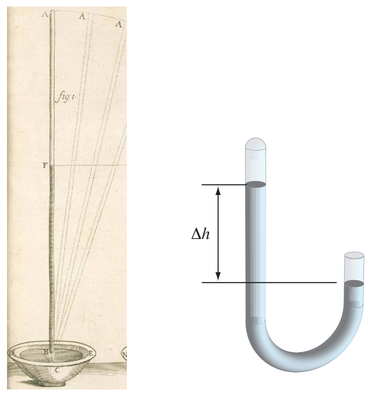

Figure 1.5: A barometer from the time of Torricelli (from the book Saggi di naturali… of the experimenters of the Accademia del Cimento in Florence, 1667). Right: A modern version of a mercury barometer built as a U-pipe with one leg open to the surrounding air.

Interpretation. Fluids at rest in the gravitational field at the surface of the Earth have the same pressure at the same height. Therefore, the pressure of the mercury (or water) at the surface of the open bowl or in the open leg of the U-pipe must be the same as the pressure of the liquid at the same height in the closed pipe. The latter is equal to the pressure of the column of liquid in the pipe. Since mercury has a density close to 14 time that of water, the mercury column in the barometer is 14 times shorter than that in an equivalent water barometer. The height of mercury columns is used to define a pressure scale in millimeters of mercury column (mmHg).

Columns of liquids in vertical pipes can be used as simple pressure gauges. Take a horizontal pipe through which water is flowing as in Fig. 1.6. If we mount an open vertical pipe onto the horizontal pipe, and if the water rises to a certain level in the vertical pressure gauge, we can use the level of water as an indication of the pressure of the water below (relative to air pressure).

Pressure differences in hydraulic circuits. Consider water in a tank having a long horizontal pipe for an outlet (Fig. 1.6). The hydraulic circuit goes from the top of the water in the tank (Point A), through the water in the tank (Points B and C), through the horizontal pipe from C to G, and finally back through the air to point A. The pressure of the liquid can be determined at points above which a water column is established (Points B, D, E and F) and at A and G (equal to air pressure p_a, which is equal to 97900 Pa for the time the data was taken; see Fig. 1.6, right).

Interpretation. Pressure can be interpreted as the hydraulic level quantity. Pressures along a path can be viewed as levels in a landscape, going up and down. When we walk along a closed path, we return to the same level. This means that all level differences add up to zero along a closed path. Pressure goes up when we go down in the liquid in the tank. It goes down in the pipe as a consequence of fluid flow friction. The drop from B to C is a consequence of the speed change of the liquid going from the wide tank to the narrow pipe.

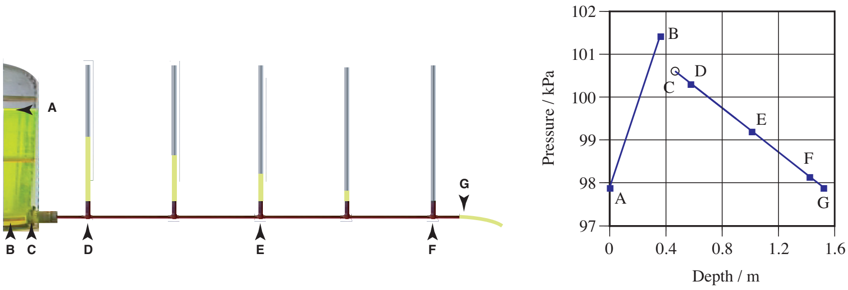

Figure 1.6: Water is flowing out of a large tank through a long, horizontal pipe. This is an example of a hydraulic circuit leading from A through B, C, …, to G, and back through the air to A. Different water levels allow for the pressure of the fluid to be measured along this circuit. Pressure of the liquid in the system at different points along a circuit, plotted as a function of position. Note the drop from B to C.

QUESTIONS

1. Why does the phenomenon shown in Fig. 1.1 show that levels of liquids rather than quantities of liquids equilibrate in communicating tanks?
2. Describe and explain the phenomenon shown in Fig. 1.1, i.e., create a word model of the system and the processes it undergoes. What concepts and ideas are used in the description and explanation.
3. Why does the photograph of the U-pipe with water and oil (Fig. 1.2) demonstrate that fluid pressures rather than levels of liquids equilibrate in communicating tanks?
4. If you take the phenomenon of equilibration in balloons as in Fig. 1.2, what does this demonstrate about the driving force for the flow of air?
5. Determine the ratios of the slopes of the straight lines in the diagrams in Fig. 1.3. Do the ratios agree with the ratios of the densities of water, olive oil, glycerine, and alcohol? What would the slope be for mercury?
6. Why is the pressure-altitude relation for the air in our atmosphere (Fig. 1.4) not linear? What does the slope of this relation at the beginning tell us about the density of air at sea level?
7. Why can we tell that there must be a pressure difference in the system in Fig. 1.6 between points B and C?
8. What is considered to be the cause of the pressure drop in the long horizontal pipe in Fig. 1.6? What is the pressure difference along the pipe if the fluid does not flow (if the pipe is plugged)?

Flows and pressure differences. When a fluid flows steadily through a pipe, we observe that its pressure drops in the direction of flow. The stronger the current, the bigger the drop. The relation between the resistive pressure drop (called the *resistive pressure difference*) and the associated volume current is called the *flow characteristic* or *resistive characteristic* (Fig. 1.7). It allows us to calculate flows if we know the associated pressure difference, or vice-versa.

The resistive flow characteristic is linear for highly viscous fluids such as oils (see Fig. 1.7, left). Water flowing through a pipe typically leads to a nonlinear characteristic (Fig. 1.7, right). It is found that linear characteristics are associated with *laminar flows* whereas the nonlinear type is the result of *turbulent flow*.

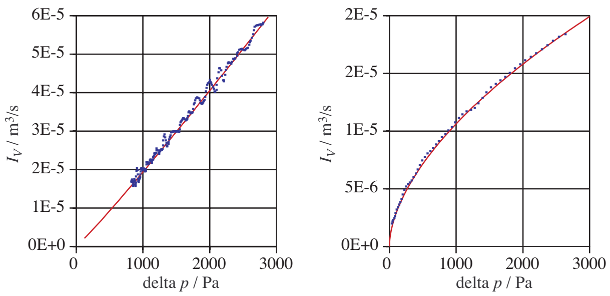

Figure 1.7: Flow characteristics for laminar flow (oil in a pipe, left) and for turbulent flow (water in a pipe, right). I_V is the symbol for volume current.

Interpretation. Fluids are viscous. Therefore, they have to be pushed through a horizontal pipe to keep flowing at a certain rate. This means that the pressure must be higher upstream, and lower downstream.

Pumps and pressure differences. Pumps set up pressure differences, and fluid flows pass through them. The relation between pump pressure difference and current of liquid is called the *pump characteristic* (Fig. 1.8). This is how continuous flow pumps are represented. Intermittently working pumps, such as the heart, may be represented by an appropriate pressure-volume characteristic.

Interpretation. A simple model of a pump is one in which the pressure difference across the pump is constant. Real pumps commonly have a more complicated type of characteristic (Fig. 1.8). The pressure difference is lower when the flows are stronger because of increasing effects of flow resistance. Put differently, a part of the pressure difference set up by the pump is needed to force the fluid through the pump. As a result, the fluid exits at a lower pressure than otherwise expected.

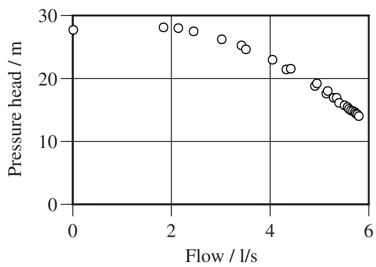

Figure 1.8: Example of a pump characteristic. In the simplest case of constant pressure difference independent of the flow, the characteristic curve would be a horizontal line. Pressure is given as the height through which water can be pumped.

The heart as an intermittent pump. Certain pumps work intermittently. They take up a quantity of liquid, increase its pressure, and emit the quantity. Then they repeat these steps. The process is represented in a pressure-volume diagram for the quantity of liquid in the pump (see Fig. 1.9 for the human heart).

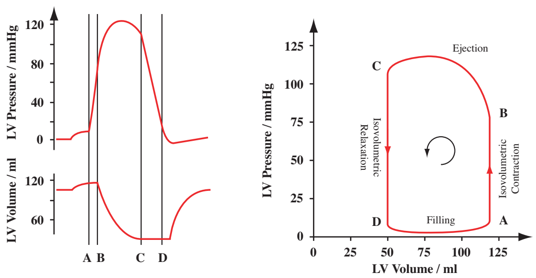

Figure 1.9: Blood pressure in left ventricle of human heart as a function of time (top left), volume of blood in left ventricle as a function of time (bottom left), and the phase plot of pressure versus volume (right). The phase plot combines two quantities which are functions of time in a single diagram (time is not explicitly visible any longer).

Interpretation. The pV-diagram of the blood in the left ventricle of the heart should be read in the direction the trajectory or path is traversed—here it is counterclockwise (Fig. 1.9). We start at the lower left corner of the almost rectangular trajectory in the phase diagram. Here, the left ventricle only contains the minimum volume of blood (called the dead volume) at very low pressure close to ambient pressure in the body. From there, the volume increases—there must be an inflow from the left atrium that receives blood from the lungs. During this filling process, the pressure of the blood remains at an almost constant low value which means that the muscle of the heart is relaxed.

When the filling is done, the valve between the atrium and the ventricle closes. Now the pressure of the blood in the ventricle increases at constant volume—meaning the heart contracts and there is no blood flow. At some point, as the heart is still contracting, the valve to the aorta opens and blood is forced out of the ventricle. The pressure first still increases before it decreases somewhat. When the dead volume is reached, the flow of blood stops (the valve closes) and the heart relaxes. As a result, the pressure of the constant volume of blood decreases to the value we started with.

The pV-phase plot shows the functioning of this pump much more clearly than volume and pressure separately as functions of time (Fig. 1.9). We can easily read the so-called stroke volume from the graph as the difference between the right and left vertical lines showing pressure changes at constant volumes. Moreover, as we will understand later (Chapter 2 and the chapter on energy), the area enclosed by the path in the pV-diagram represents the energy of the pumping heart delivered to the flowing fluid.

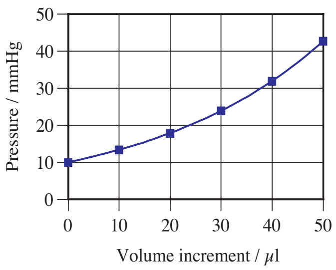

Figure 1.10: Pressure in the living human eye as a function of added volume of liquid. Note the non-standard units. The slope of the p-V relation is called the elastance of the eye (diagram on the right). The diagram is called the capacitive characteristic.

Fluid containers and pressure differences. When fluids (liquids and gases) are filled into tanks, the pressure is higher if there is more of the fluid in the tank. We know this from air balloons, from the open tanks seen above, from medicine, and many other examples.

An important medical example is the eye. High pressure is detrimental, so it is important to understand the effects of changes of the volume of liquid in the eye. The pressure rises with volume in a manner similar to what we can expect for pressure vessels such as balloons or the kinds used in engineering (Fig. 1.10). The pressure rises more steeply for larger volumes. The relation between fluid pressure and volume stored is called the *capacitive characteristic* of the storage element.

Interpretation. In the case of an open tank (or a lake or the ocean), the added weight of the added fluid leads to a rising pressure. In pressure vessels (membrane accumulators used in hydraulic systems, the eye, the skull with a brain, the aorta of the blood circulatory system), the elastic container wall is stressed more strongly if there is more fluid in the storage element. Moreover, typical walls get stiffer when stressed more strongly which explains the steeper rise of the pressure with increasing volume.

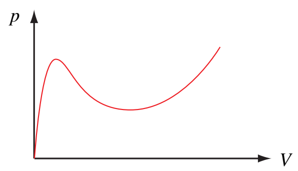

Figure 1.11: Qualitative sketch of the pressure-volume relation of a typical toy rubber balloon.

Rubber balloons are an exception to the rule that the pressure rises with increasing volume. Rubber has an unexpected property: At small volumes, the pressure indeed rises with increasing volume. However, there is a range of volumes when the pressure drops with increasing volume before rising again (Fig. 1.11).

Compressibility of fluids and the pressure-volume relation of air. Liquids and gases can be compressed by increasing the pressure. As we know from experience, this effect is very small for liquids but quite noticeable for gases. In fact, we will often treat liquids as incompressible, but we cannot ignore the compressibility of gases.

When a quantity of gas such as air is enclosed in a chamber whose volume is slowly decreased, the pressure of the gas rises. We know from the operation of bicycle pumps that the compressed air can get quite hot. To remove the effect of still another quantity, i.e., temperature, we change the volume slowly enough for the temperature to stay constant.

The pressure of air rises in a simple manner with decreasing volume if the temperature is kept constant. The result of one such measurement is shown in Fig. 1.12. The data seem to trace a hyperbola, and indeed this function fits data well for simple gases at relatively low pressures and high enough temperatures (which is to say at relatively low densities).

Interpretation. The effect of the rise of pressure with decreasing volume is interpreted as the result of the elasticity of the material. We know, for example, that we have to increase the pressure on a metal spring to compress it more strongly.

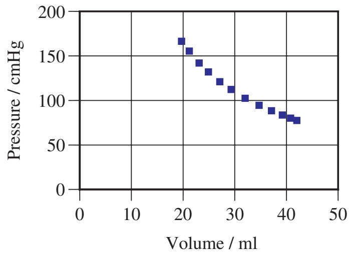

Figure 1.12: Pressure versus volume of a quantity of air. Pressure and volume were recorded for constant temperature of the air.

QUESTIONS

9. Consider the type of flow characteristic found for oil flowing in a pipe as in Fig. 1.7 (in the left diagram). What do you expect the characteristic curve to be for a pipe twice as long as the one used in the experiment? What do you think the characteristic curve should be if the pipe has a larger radius?
10. Sketch a characteristic diagram of an ideal (continuous flow) pump.
11. Investigate the operation of the heart of a human and sketch the pressure-volume diagram of blood in the left of right ventricle of such a heart.
12. What does the pressure-volume relation of a cylindrical tank with a liquid stored in it look like? What is the relation for a cylindrical tank with a smaller diameter than that of the first tank?
13. Imagine a spherical tank for liquids. The tank is open to the air. What would be the pressure-volume characteristic diagram of this storage device?
14. The aorta of a human (or of a mammal, for that matter) is an elastic storage device for blood. Sketch a pressure-volume relation for blood in the aorta. How does the relation of an older person compare to that of a younger one?
15. Use the qualitative sketch of the pressure-volume relation of a toy balloon (Fig. 1.11) to explain why the air can flow from a less inflated balloon to a more inflated one. (Remember that we would normally assume the more inflated balloon to have the higher pressure.) Is it possible for the air to go from the more inflated one to the less inflated one?

Charging and discharging single tanks. A straight-walled tank is filled with oil. The oil drains through a horizontal pipe fitted at the bottom of the tank. We observe how the level of oil decreases as a function of time (Fig. 1.13, diagram at center). The measured values closely fit an exponentially decaying function of time.

If a pump is fitted to the end of the pipe, the tank can be filled with oil instead. If the pump sets up a constant pressure difference (irrespective of the magnitude of the flow), the filling curves are exponentials again (Fig. 1.13, diagram at right).

Interpretation. The outflow is driven by the pressure difference across the pipe which is equal to the pressure difference across the tank. Since the latter is proportional to the level of oil, and since the flow (which is laminar) is proportional to the pressure difference, the flow is proportional to the level. The flow defines the rate of change of oil in the tank. So, in summary, the rate of change of the level of oil is proportional to the level of oil. This leads to the functions observed.

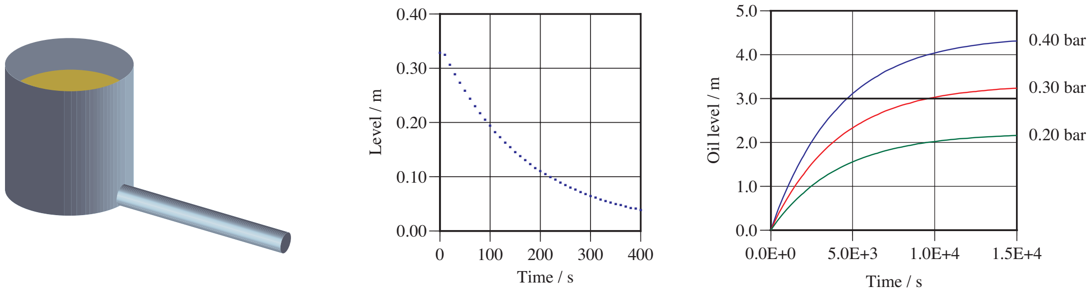

Figure 1.13: A straight-walled tank filled with oil is discharged through a horizontal pipe at the bottom (center diagram) or charged with the help of a pump (diagram on the right). The measured levels closely fit exponential functions of time.

Changing flow speeds. When a fluid enters a region where it flows faster than in the region it comes from, its pressure drops. When its enters a region where it flows more slowly, the pressure rises. Changing flow speeds are the result of narrowing or widening of the channel through which the fluid flows.

Interpretation. Consider water flowing from a wide tank into a thin pipe, or blood flowing from the heart into the aorta. The same flow must pass through a narrower conduit than before. Liquids are incompressible, so they must speed up.

To make a fluid speed up from one point to another along its path, it must be pushed from behind (like we have to push bodies so they speed up). So the pressure is higher where the fluid comes from. In the case of a decrease of flow speed, the pressure is lower from where the fluid comes.

Flow through chains of tanks. Five relatively narrow tanks are connected at their bottoms by hoses. The first is tank filled to a high level, the second and all the others are basically empty at start. When the pipes are opened, water begins to flow from left to right (Fig. 1.14), and the levels change as seen in the graph. This is an example of a substance travelling through storage elements as it occurs in diffusion (see Chapter 4).

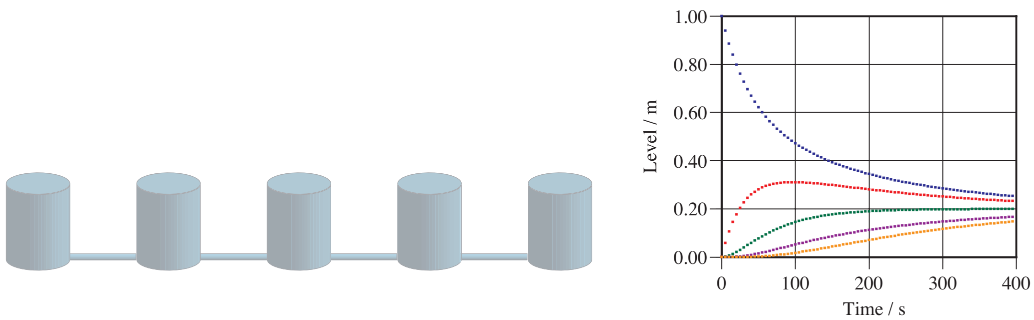

Figure 1.14: A chain of tanks filled with water. Water stands at different levels at first (the four tanks on the right are virtually empty at first). When the pipes are opened, water levels change with time.

Interpretation. For water to flow from tank to tank, there must be appropriate pressure differences across the pipes. High pressure differences let the levels in neighboring tanks change fast. Since the tanks on the right have the same initial levels, their levels do not rise initially. It takes a while for them to rise as well. All the levels tend to equilibrate. Toward the end, the flows have become small and the processes run slowly.

QUESTIONS

16. Describe the process of draining of a tank such as the one in Fig. 1.13. Produce a word model for the process.
17. Consider the diagram of discharging of an oil tank shown in the middle of Fig. 1.13. What would be the curve showing discharging of a tank that has twice the cross section? What would the curve be for a pipe having half the length?
18. Why don't the levels in the tanks on the right in Fig. 1.14 change right from the beginning?

1.2 QUANTITIES AND MATHEMATICAL OPERATIONS

We need three primitive system and process quantities to describe and explain hydraulic phenomena. One is for amounts of fluids stored in systems, the second for flows, and the last is for the pressure at a point of a fluid. On the basis of these, related ones are defined by mathematical procedures. Properties of systems and elements—such as resistance and capacitance—are introduced together with special laws found to hold for the particular system (see Section 1.6).

1.2.1 Primitives

Primitives are terms or quantities that cannot be defined on the basis of other quantities. They are fundamental and are taken from everyday notions and mental images of what we see happening around us.

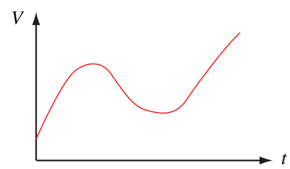

Figure 1.15: Volume as a function of time. Note how it changes: changes may be positive or negative, slow or fast.

Quantity of fluid stored. There are three possible measures of amounts of fluids stored in systems: Volume, mass, and amount of substance. The volume of liquids is chosen to measure their amount. In processes, the volume of a liquid stored is a function of time (Fig. 1.15). The unit of volume is m^3. We use the symbol V to denote the volume of a fluid stored in a system. Mass and amount of substance are related to volume:

$$m = \rho V \quad , \quad n = \frac{1}{M_0} m \tag{1.1}$$

m is its *mass* measured in kg, ρ is the *density* (in kg/m^3), and n stands for amount of substance (unit: mole). M_0 is the *molar mass* of the substance.

Volume current. The volume current (or volume flux, or simply flow) describes the flow of a fluid through pipes or other channels. The current allows us to calculate how much fluid is transported in a given period of time. The unit of volume current is m^3/s. We use the symbol I_V to denote a flow of fluid, i.e., a volume current. In processes, the flow is a function of time (Fig. 1.16).

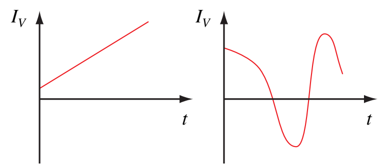

Figure 1.16: Volume currents as a functions of time. Currents can change slowly or quickly, they can be positive or negative.

Pressure. The pressure measures the state of the fluid which is directly responsible for hydraulic processes. Its unit is Pa (Pascal), the symbol used is p. 10^5 Pa equals one bar (1 bar). The pressure at the bottom of a 10 m high column of water here on Earth is about 1 bar. So is the average air pressure at sea level.

QUESTIONS

19. Assume the tank used in the process shown in Fig. 1.13 to have a cross section of 0.50 m^2. Convert the diagram (in the middle) to one showing the volume of liquid as a function of time.
20. Determine the amount of substance of 1 m^3 of water.
21. Sketch a current that is constant at first, and then decreases at a constant rate.
22. Describe the current shown in the diagram on the right in Fig. 1.16.

1.2.2 Derived quantities

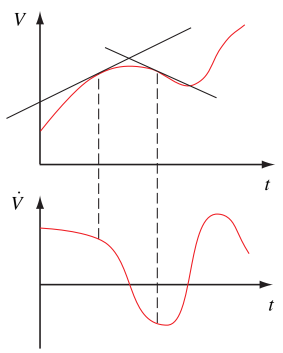

Figure 1.17: Determining the rate of change of volume by calculating the slope of tangents to the *V*(*t*) curve. The symbol *V* with the dot above it denotes the rate of change of *V*.

Change of volume. The change of volume over a period of time is defined as the difference between the later and the earlier values:

$$\Delta V_{1\to 2} = V_2 - V_1 \qquad (1.2)$$

The change of volumes can be read from graphs (on the vertical axis) and from tables, or it can be calculated from formal representations of the functions.

Rate of change of volume. The rate of change of volume describes how fast the volume stored changes (Fig. 1.17). Symbols are dV/dt or V with a dot above it (pronounced V-dot). The unit of rate of change is m^3/s. The rate of change is determined graphically from the slope of tangents at points of the curve in the volume-time diagram.

Transported volume. If we know a volume current as a function of time (Fig. 1.16), we can determine the volume of fluid transported or exchanged with this flow (Fig. 1.18, symbol V_e). Graphically, the transported volume corresponds to areas between the function $I_V(t)$ and the time axis for an interval of time from, say, t_1 to t_2. The mathematical operation is called integration of the current over time:

$$V_e = \int_{t_1}^{t_2} I_V(t)\,dt \qquad (1.3)$$

In system dynamics tools, this computation is performed automatically if one represents the current by a flow symbol connected to a storage element (Fig. 1.18, right). The storage element obtains the integrated quantity. Integration can be performed graphically by hand or numerically in spread sheets using the operations of multiplication and addition only.

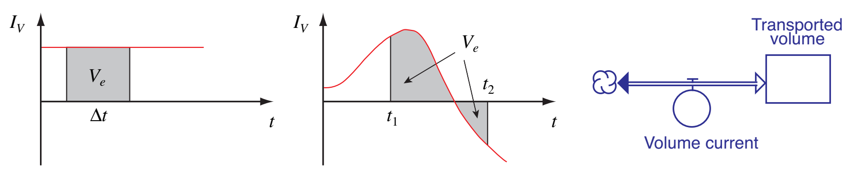

Figure 1.18: The volume transported with a volume current is determined by calculating the area between the $I_V(t)$ curve and the *t*-axis. The same operation in a system dynamics model (right).

From rate of change of volume to volume. If the rate of change of volume stored in a system is known, we can calculate the change of the volume resulting during a period of time (Fig. 1.19). As in the case of the computation of transported or exchanged quantities (as a consequence of a flow), the mathematical procedure is that of integration, in this case of the rate of change of volume over time:

$$\Delta V = \int_{t_1}^{t_2} \dot{V}(t)\,dt \tag{1.4}$$

If we also know the initial value of the volume, we can calculate the volume as a function of time (Fig. 1.20). This can be done graphically or numerically (in a spread sheet, or with the help of a system dynamics tool).

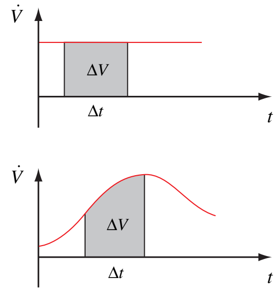

Figure 1.19: Changes of volume calculated from the rate of change.

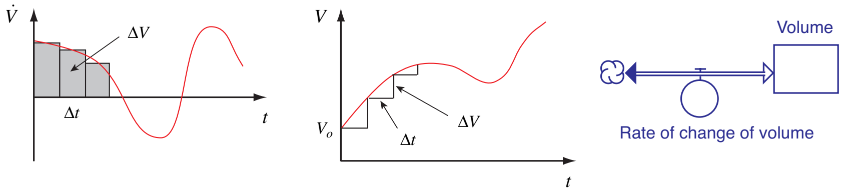

Figure 1.20: The volume is found from the rate of change of the volume and the initial value of the volume by (numerical) integration. Right: Integrator in Stella.

Pressure difference. The pressure difference is the difference of pressures in a fluid at two different points A and B, independent of the physical reasons for the pressure difference. Usually, it is defined as downstream value minus upstream value, or simply later minus earlier in a chosen direction:

$$\Delta p_{AB} = p_B - p_A \tag{1.5}$$

QUESTIONS

23. The volume of water in a tank increases quickly from 10 liters (Point 1) to 20 liters (Point 2). Then it decreases slowly to 15 liters (Point 3). What is the change of volume from Point 1 to Point 2? From 2 to 3? From 1 to 3?
24. The volume of water in a tank changes linearly from 20 liters to 12 liters in 10 s. What is the rate of change of volume at time 5 s? What is the average rate of change for the entire period?
25. Are there points where the rate of change of volume is equal to zero for the volume sketched in Fig. 1.17? If so, what is the property of those points?
26. A current of liquid is constant. Write the equation for the volume transported with this current for a chosen interval of time Δt.
27. A current of water changes linearly from 10 liters/s to 4 liters/s in 20 s. How much water has been transported during these 20 s?
28. The rate of change of volume of a liquid in a container equals – 2.5 m^3/s. By how much does the volume change in 2 minutes?
29. Explain the procedure for calculating the volume of liquid in a tank from its rate of change. What other information is needed to perform this calculation?

1.3 SYSTEMS ANALYSIS I: LAWS OF BALANCE

A part of systems analysis consists of a number of fairly well defined steps which lead to an overview of the processes occurring in a system, and to the formulation of the laws of balance associated with systems and processes.

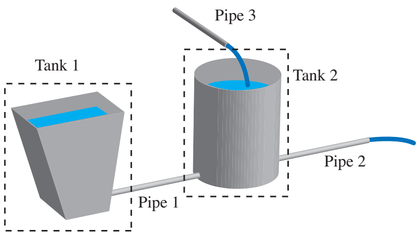

Figure 1.21: A situation sketch is part realistic, part abstract. It shows the system in its environment and may contain details concerning system sizes, etc.

1. Draw a situation sketch. Create a sketch of the system to be investigated, and its environment. The sketch should be half realistic, and can contain information about the sizing of the system and its parts (Fig. 1.21).

2. Choose systems or elements. Choose one or several systems, subsystems, or elements. A system is either an identifiable body or a region of space (in the latter case it is called a control volume). Here we choose two control volumes containing the tanks in Fig. 1.22. Generally, there is more than one way of choosing elements. (The word *system* is used differently here than in systems science.)

3. Choose quantities to be accounted for. In hydraulics, the quantities to be accounted for are amounts of fluid (volumes).

4. Cut systems or elements out of environment and identify processes. Draw abstract representations of the systems chosen (such as the dashed rectangles in Fig. 1.22). Identify all the flows since these represent the processes (to find them, ask why there are processes and "walk around" the system to find inputs and outputs). Draw an arrow for each flow with respect to its system. Label the arrows. Enter the stored quantities (Fig. 1.22). (Step 4 is called creating *free body diagrams*.)

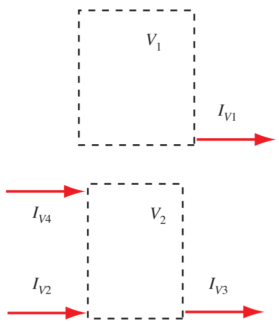

Figure 1.22: A "free body diagram" for two subsystems. Processes and stored quantities are identified for each subsystem.

5. Formulate laws of balance. For each of the subsystems and for each of the stored quantities, a law of balance is formulated (normally in dynamical or *instantaneous form*). A law of balance relates all processes to how fast the system content changes:

$$\begin{aligned} \dot{V}_1 &= I_{V1} \\ \dot{V}_2 &= I_{V2} + I_{V3} + I_{V4} \end{aligned} \tag{1.6}$$

If the laws are to be formulated for a certain period, the equations are written in *integrated form* that relates changes of stored quantities to all transported or exchanged amounts:

$$\begin{aligned} \Delta V_1 &= V_{e1} \\ \Delta V_2 &= V_{e2} + V_{e3} + V_{e4} \end{aligned} \tag{1.7}$$

In system dynamics diagrams, laws of balance in their instantaneous forms can be represented graphically by combinations of storage and flow symbols (Fig. 1.23).

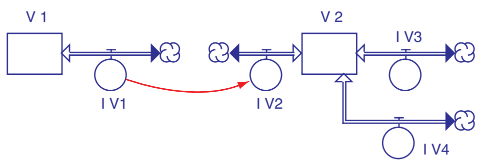

Figure 1.23: Representation of laws of balance in system dynamics diagrams. The thin arrow denotes the interaction rule (action and reaction).

6. Formulate interaction rule. In our example, there is a single interaction: The flow out of Tank 1 equals the flow entering Tank 2 (see Fig. 1.23):

$$I_{V1} = -I_{V2} \tag{1.8}$$

Questions

30. A tank has neither inlets nor outlets for water. Express both the instantaneous and the integrated forms of the law of balance of volume of water for this tank.
31. There is an inflow of liquid to a tank. At the same time, liquid is flowing out. The outflow is equal in magnitude to the inflow. What is the law of balance of volume for this situation?
32. A pipe is branching. What is the relation between the three currents related to the junction? How does this junction rule follow from the law of balance of volume in Equ. 1.6?
33. Why is it important to clearly identify the systems or elements for which a law of balance is to be written? Explain what can go wrong if this is not done carefully.
34. During a period, 100 liters of water flowed out of a tank. At the same time, 20 liters flowed in through a pipe. During the same period, the volume of water changed by + 30 liters. Explain how this is possible.
35. Explain the meaning of the interaction rule.
36. If there are three tanks exchanging a liquid, how many times does the interaction rule apply?
37. What happens to the law of interaction formulated in Fig. 1.23 if the two tanks are treated as a single element?

1.4 Systems Analysis II: Pressure and Pressure Differences

Pressure takes the role of the hydraulic level. Pressure differences are level differences which we consider to be the causes for hydraulic processes. Alternatively, we may look at processes leading to pressure differences.

Pressure differences in closed hydraulic circuits. The pressure changes from point to point in a closed hydraulic circuit. To make use of this observation, choose a few important points in the system (usually at the inlets and outlets of elements such as pipes, pumps, and tanks). Label pressure differences from point to point (Fig. 1.24) by introducing arrows and symbols Δp_{AB}, etc.

We can draw a diagram looking like a "landscape" (Fig. 1.24). When we are back at the origin, the pressure is the same. Therefore, the sum of all pressure differences in a closed circuit must be equal to zero (loop rule, Kirchhoff's Second Law):

$$\Delta p_{AB} + \Delta p_{BC} + \Delta p_{CD} + \ldots = 0 \tag{1.9}$$

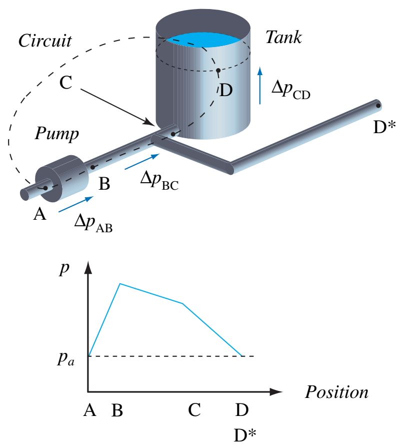

Figure 1.24: Pressures in a fluid system form a kind of "pressure landscape" or "hydraulic landscape" which shows ups and downs.

Pressure differences and processes. Pressure differences are associated with many different types of processes and systems (see Section 1.6):

- fluids stored in pressure vessels
- fluids stored in tanks, height differences
- pumps and turbines
- flow resistance
- differences of flow speed (at different points along a streamline, because of changes of cross sections)
- changing flows (changes in time will be discussed in a later chapter).

Processes in an element. Consider elements which are between two chosen points in Fig. 1.24 (example: pump and pipe from point A to point C, Fig. 1.25). Identify all possible processes occurring in this element. Each of the processes shares a part of the pressure difference Δp_{AC}:

$$\Delta p_{AC} = \Delta p_{\text{process 1}} + \Delta p_{\text{process 2}} + \dots \qquad \textbf{(1.10)}$$

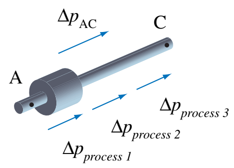

Figure 1.25: A "free body diagram" of a subsystem for identifying pressure differences. The pressure difference from A to C is shared by several processes.

QUESTIONS

38. Trace the pressure of a water in the following system. Water is taken from a lake at a certain depth. It is pumped through a horizontal pipe on land by a first pump. After a certain distance, a second pump is used, and the water is pumped into a lake higher up on a hill.
39. Oil is pumped around in a hydraulic circuit having a pump, an engine, and some pipes. If the pressure difference across the engine is 2.0 bar, what is the pressure difference across the pump?
40. Take a point in time when the blood leaving the left ventricle of a heart has a pressure of 130 mmHg. The pressure difference across the arteries and smaller vessels through the body is 70 mmHg. Before the right ventricle, the pressure of the blood is 20 mmHg. How is this possible?
41. Oil flows through a pipe leading upward. What processes lead to the pressure difference across the pipe in the direction of flow? What is the sign of the pressure change in the direction of flow?

1.5 FLUID PROPERTIES

There are a few fluid properties that must be taken into account when explaining phenomena such as storage or flow. The properties important to us at this point are density, viscosity, and compressibility.

1.5.1 Density

Density was already introduced in Equ. 1.1 as the factor relating mass and volume. Since mass is taken as the source of the phenomenon of gravity, and since gravity plays a role in the vertical pressure profile in fluids in a gravitational field (Fig. 1.3 to Fig. 1.6), density is—among other things—related to pressure differences in stored fluids or in fluids flowing up or down in pipes (Section 1.6.5).

1.5.2 Viscosity

Viscosity is the property of fluids that leads to flow resistance. It is quite clear that the tank in Fig. 1.13 drains more quickly if the oil is less viscous, and more slowly if the liquid is more viscous. Castor oil, for example, is much more viscous than olive oil, and olive oil is much more viscous than water. Therefore, viscosity critically influences the type of flow. Water flowing out of the tank in Fig. 1.13 would exhibit turbulent flow whereas the flow of oil through the same pipe under similar conditions would be laminar.

Viscosity is said to be the cause of fluid friction between layers of fluid that move past each other (this is called shear stress). Consider oil between two horizontal plates (Fig. 1.26). If the upper plate is pulled to the right, the layers of oil move at different speeds. The oil sticks to the surfaces of the plates, so the oil speed is highest at the top (equal to the speed of the upper plate) and zero at the bottom.

The quantity of motion (momentum) imparted to the upper plate leaves this plate (otherwise it would speed up continually) and flows down through the layers of oil to the lower plate. The quantity of momentum flowing through the oil every second and per square meter (j_p, unit: Pa) depends upon the gradient of speed in the y direction (dv/dy):

$$j_p = \eta \frac{dv}{dy} \tag{1.11}$$

The factor relating the transport of momentum and the gradient of speed depends upon how viscous the oil is, and is called the viscosity (η).

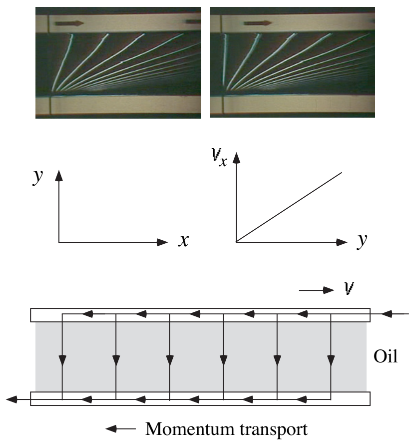

Figure 1.26: Viscosity is the factor relating sheer stress in a fluid (j_p) and the gradient of speed perpendicular to the motion of fluid layers (dv/dy). The photographs at the top show bubbles in oil that is moving as the result of the motion of the upper plate.

1.5.3 Compressibility and the pressure-volume relation for air

Fluids are compressible, meaning their volume changes if the pressure is increased (normally, the volume decreases). In general, the compressibility of liquids is low—so we typically assume them to be incompressible. Gases, however, show marked compressibility (see Fig. 1.27).

The compressibility of a fluid is defined as the relative change of volume divided by the change of pressure causing the change of volume:

$$\beta = -\frac{1}{V}\frac{dV}{dp} \tag{1.12}$$

The term dV/dp is the inverse of the slope of the tangent to the pV-relation in the pV-diagram (Fig. 1.27). One has to add some more information to the definition such as whether or not the temperature of the fluid was kept constant during compression (which was the case for the data in Fig. 1.27), or if other conditions applied (such as no heating or cooling; this will be studied in Chapter 3).

For air and other gases at low pressures and high (constant) temperatures, the pV-relation is a simple hyperbola, meaning that the product of pressure and volume is constant

$$pV = \text{const.} \tag{1.13}$$

if T = const. (T is the temperature). This means that the compressibility of air is equal to $1/p$. Air becomes harder to compress if it is already compressed.

The constant relating pressure and volume of air at constant temperature depends upon temperature and upon the quantity of the gas used in the experiment. The former property will be explored in Chapter 3. The latter can be made clear quite easily. Double the quantity of air will take double the volume at given pressure (and

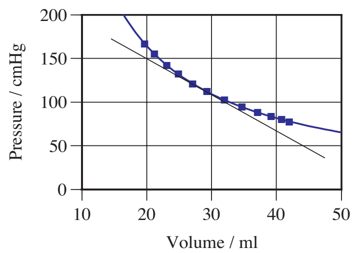

Figure 1.27: Pressure-volume relation for air. The inverse of the slope of the tangent is a measure of the compressibility of the gas.

temperature), and double the quantity will double the pressure at fixed volume:

$$pV = cn \tag{1.14}$$

n is the amount of substance of the gas, and the factor c depends upon the temperature of the fluid.

1.6 Constitutive Laws: Storage, Pumps, and Flow

Constitutive laws, or special laws, are relations that depend upon the types of elements and fluids used in a system, and upon circumstances. In contrast to the laws of balance or the loop rule, they are not general. They describe the peculiarities of processes and objects. Much of the work in physics goes into constructing useful constitutive laws.

There are constitutive laws covering all different types of processes. In particular, there is at least one special law for each particular pressure difference occurring in a system.

1.6.1 Storage of fluids

Storage elements are responsible for the dynamics found in (hydraulic) systems. They work by providing a relation between amounts of stored fluid and the pressure difference set up across them. Storage elements are pressure vessels (such as diaphragm accumulators, the heart, the skull with the brain, etc.) or containers that "stack" fluids in the gravitational field.

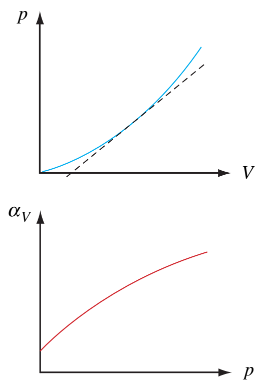

Figure 1.28: Pressure as a function of stored volume (top). The slope of the characteristic curve is called the elastance α_V of the storage element. Typically, the elastance is a function of pressure.

Capacitive characteristic. If fluids are stored in tanks or pressure vessels, the pressure difference increases with increasing amount of stored fluid. In other words, there is a relation between the volume stored and the associated pressure difference (which we call a capacitive pressure difference Δp_C). The relation is called a capacitive characteristic (Fig. 1.28). Examples are shown in Fig. 1.29.

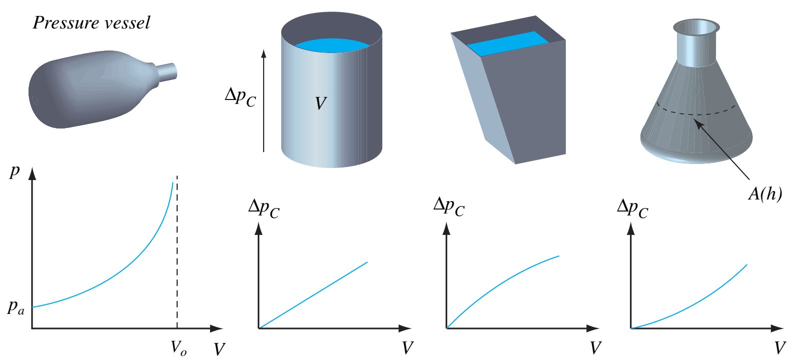

Figure 1.29: Different containers have different capacitive characteristics. The hydraulic capacitance of an open container is proportional to the (variable) cross section.

Elastance and hydraulic capacitance. The characteristic can be expressed mathematically if we introduce the elastance α_V, i.e., the factor which tells us how easy it is to increase the pressure with a given amount of fluid:

$$\dot{p}_C = \alpha_V \dot{V}$$
$$\Delta p_C = \alpha_V \Delta V \quad \text{if} \quad \alpha_V = \text{const.} \tag{1.15}$$

α_V is equal to the slope of a tangent to the characteristic curve (Fig. 1.28). This means that the elastance measures the stiffness of container walls (in the case of pressure vessels) or the inverse of the cross section of a tank. The unit of elastance is Pa/m^3.

Alternatively, we can introduce the hydraulic capacitance C_V (units m^3/Pa) which is defined as the inverse of the elastance ($C_V = 1/\alpha_V$):

$$\dot{V} = C_V \dot{p}_C$$
$$\Delta V = C_V \Delta p_C \quad \text{if} \quad C_V = \text{const.} \tag{1.16}$$

For a liquid of density ρ in an open container the capacitance is

$$C_V = A(h)/(\rho g) \tag{1.17}$$

Equ. 1.16 suggests a way to determine volume changes from pressure changes if the capacitance is known (as a function of pressure). For constant capacitance, we simply multiply the pressure difference by the capacitance. Geometrically, this corresponds to the calculation of the area of a rectangle. This tells us that in general the change of volume associated with a change of pressure is equal to the area between the capacitance – pressure function and the pressure axis (Fig. 1.30).

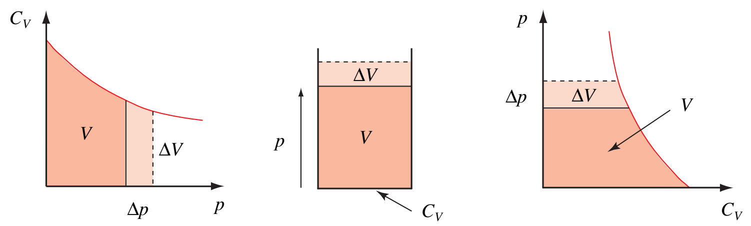

Figure 1.30: The change of volume related to a change of pressure is obtained from integrating the capacitance over pressure (left). This can be interpreted more intuitively by considering the diagram as a type of tank with a fluid contained in it (center and right).

If rubber is used for a vessel—such as in a toy balloon—the pV-characteristic is much more complex than what we discussed here, indicating that we have to be careful with the definition of a capacitance or elastance. For the part of the characteristic curve in Fig. 1.11 where the pressure drops with increasing volume, the usual definition leads to a negative capacitance which does not make much sense. In such cases, one should directly work with the pV-characteristic rather than introduce the elastance or capacitance.

Questions

42. For a given situation of oil between two plates (Fig. 1.26), how strongly does one have to pull if one wants to move the upper plate twice as fast? Compare this relation to the flow characteristic of oil through a pipe (Fig. 1.7).
43. What happens to the pressure of a fixed quantity of air if its volume is reduced to half the initial value (at constant temperature)?

44. Describe the everyday experience that tells us that the compressibility of air decreases with increasing pressure.
45. Why is the second characteristic relation shown in Fig. 1.29 linear?
46. Why do the first and the last of the containers shown in Fig. 1.29 have capacitive characteristic curves that rise more steeply for bigger volumes of liquids stored?
47. Explain the meaning of elastance and hydraulic capacitance.
48. Sketch the capacitive characteristic relations for two cylindrical tanks, one having a large cross section, the other having a small cross section.
49. What happens to the capacitance of the human aorta as a person gets older?
50. Explain how to calculate the change of volume for a container having a nonlinear characteristic relation.

1.6.2 Pumps

Pumps come in many different types and forms, ranging from the heart to microengineered or large industrial pumps. Here we are only interested in their overall performance.

Process diagram. Pumps make fluids flow, and they increase their pressure. This simple fact is best represented in a process diagram of the type shown in Fig. 1.31. Process diagrams represent a system (or an element of a system) and show what happens with the basic quantities (here: volume current and pressure) used to describe a hydraulic process. The process is depicted as a kind of inverse water fall.

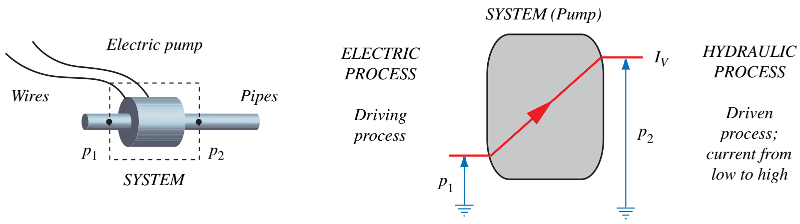

Figure 1.31: As a fluid is forced through a pump, its pressure is made to go up. Figuratively speaking, the fluid current is forced uphill.

Pump characteristics. Pumps set up pressure differences, and fluid flows pass through them. Therefore, we define their operation by describing the relation between these quantities. A simple model of a continuous flow pump is one in which the pressure difference Δp_P is constant. Real pumps commonly have a more complicated type of characteristic which also depends upon the efficiency (Fig. 1.8). Pressure-volume diagrams are used to represent the operation of intermittently acting pumps such as the heart (see Fig. 1.9).

The process diagram used to describe the operation of a pump can be used to introduce the notion of the energy delivered to the fluid by the pump (Chapter 2).

1.6.3 Resistive fluid flow

When fluids flow through pipes, their pressure drops in the direction of flow because of fluid friction. This pressure drop is called a resistive pressure difference Δp_R, and it is characteristic of the flow which, in turn, depends upon fluid properties and pipe dimensions.

Process diagram. Since the fluid goes from high to low pressure (different from what it does in a pump; Fig. 1.31), we say that it is a driving process. We know that in resistive flow, the process caused by the flow of fluid consists of the production of heat (Fig. 1.32).

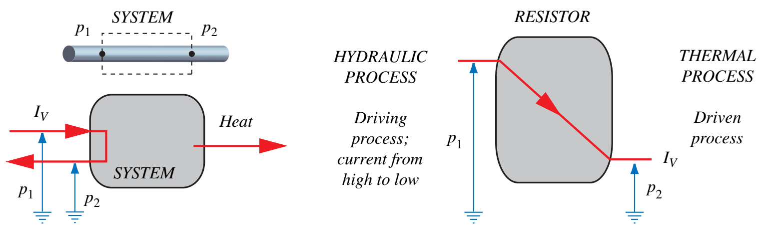

Figure 1.32: Process diagram of resistive fluid flow. The flow element may be called a resistor. Here, the driving process is the flow. The driven process consists of the production of heat. Right: Waterfall representation.

Flow characteristic. The relation between the resistive pressure drop Δp_R and the associated volume current is called the flow characteristic (Fig. 1.7). It allows us to calculate flows if we know the associated pressure difference, or vice-versa. There are two types of flow (laminar and turbulent) leading to two different characteristic curves.

Types of flow and flow speed. There are two types of flow: laminar and turbulent. Usually, turbulence sets in when a combination of flow speed, fluid viscosity, density, and pipe dimensions called the *Reynolds number* reaches a critical value. The Reynolds number is defined by Re = $\rho v D/\eta$. In the case of a pipe, D denotes the diameter. η and ρ are the viscosity and the density of the fluid, respectively. In smooth circular pipes, the transition from laminar to turbulent flow occurs at Re = 2300 or below.

The fluid speed is related to the volume flux and the cross section A of the conduit:

$$I_V = Av \tag{1.18}$$

Laminar Flow. For laminar flow, the characteristic relation is linear. In this case, we can write the flow law with the help of a hydraulic conductance G_V (units $m^3/(s\cdot Pa)$) or its inverse, the hydraulic resistance R_V (units $Pa\cdot s/m^3$):

$$I_V = G_V \Delta p_R \quad \text{or} \quad I_V = \frac{1}{R_V}\Delta p_R \tag{1.19}$$

There is an expression for the hydraulic conductance or resistance for laminar flow in pipes with circular cross section which is called the law of Hagen and Poiseuille:

$$R_V = \frac{8\eta l}{\pi r^4} \tag{1.20}$$

r and l are the radius and length of the pipe, η is the viscosity of the fluid. The viscosity of a fluid tells us how "thick" it is.

Turbulent flow. In turbulent flow, the flow increases less rapidly with an increase of the associated pressure difference (diagram on the right of Fig. 1.7). The turbulent characteristic function is close to the square root function for many practical cases. Therefore:

$$I_V = k\sqrt{\Delta p_R} \quad \textbf{(1.21)}$$

This simple relation suffices as a first approximation. k is called the turbulent flow factor. This factor is similar to a conductance, however, the terms resistance and conductance are only used for laminar flow.

1.6.4 Changing fluid speed

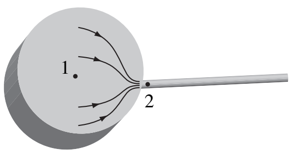

Figure 1.33: A tank and pipe viewed from above. The fluid enters a small cross section from the wide cross section of the tank.

When fluids flow through channels having varying cross sections, the flow speed changes from point to point along a flow line (Fig. 1.33). There is a pressure difference (called the Bernoulli pressure difference) associated with this change (the pressure decreases if the flow speed increases):

$$\Delta p_B = -\frac{1}{2}\rho\left(v_2^2 - v_1^2\right) \quad \textbf{(1.22)}$$

Speed increases if the fluid presses harder from behind than from the front.

QUESTIONS

51. What does the process diagram of a pump depict? Compare it to the process diagram of resistive flow of a fluid through a pipe.
52. Consider the pump characteristic in Fig. 1.8. How high can this device pump water if the flow is to be maintained at 4 liters/s?
53. We wish to pump water some 40 meters high. Consider pumps whose characteristic is shown in Fig. 1.8. What could you do to achieve your aim? How big could the flow of water be made?
54. Water is to be pumped through a horizontal pipe with the help of a pump whose characteristic looks like the one shown in Fig. 1.8. The flow of water through the particular pipe has a characteristic relation like the one seen in Fig. 1.7 (on the right). How can you use the characteristics to graphically determine the magnitude of the current of water established by the pump?
55. How could you use oil flow through a horizontal pipe to determine the viscosity of the oil?
56. By how much does a turbulent flow change when you double the pressure difference?
57. Is the pressure change in Equ. 1.22 a temporal change or a spatial change?
58. Explain why there should be a pressure difference associated with the situation represented in Fig. 1.33. What is the process associated with the pressure change?
59. When blood flows out of the left ventricle of the heart into the aorta, does its pressure change only because of flow resistance?

1.6.5 Gravity and height differences

Hydrostatic pressure. For fluids which are "stacked" in a gravitational field, i.e., systems where the weight of the fluid is responsible for a pressure difference, there is a simple relation between pressure difference and height difference (Fig. 1.34). It can be derived from the observations in Fig. 1.3 which are summarized in Fig. 1.35:

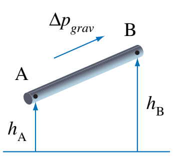

Figure 1.34: Going uphill in a fluid "stacked" in the gravitational field, the pressure decreases.

$$\Delta p_{\mathrm{Grav}} = -\rho g\left(h_{\mathrm{B}} - h_{\mathrm{A}}\right) \quad \textbf{(1.23)}$$

This relation can be used to calculate pressure differences (and capacitive characteristics) for fluid tanks (Fig. 1.29). It is correct for constant density only.

Pressure gradients. According to the example of hydrostatic pressure in an incompressible liquid (Equ. 1.23), the pressure gradient in the upward direction is

$$\frac{dp}{dh} = -\rho g \tag{1.24}$$

As observed in Fig. 1.3, the pressure gradient is proportional to the density of the liquid (see also Fig. 1.35). Furthermore, it must depend upon the strength of gravity (g). The negative sign tells us that the pressure decreases if we go upward.

Pressure in the Earth's atmosphere. The pressure-height relation for the atmosphere is not linear (Fig. 1.4). This is so since the fluid is a gas whose density changes with pressure (and temperature). Nevertheless, the expression for the vertical pressure gradient is the same as that for incompressible fluids, i.e., Equ. 1.24 still holds.

If we know how the density of the air depends upon pressure and temperature, the relation for hydrostatic equilibrium (Equ. 1.24) can be solved. Even though this is not realistic, one often considers the case of an *isothermal* atmosphere (an atmosphere where the temperature does not change in the vertical direction). If this is the case, the density is proportional to the pressure (see Equ. 1.13 and Chapter 3) leading to an exponential pressure-height relation:

$$p(h) = p(0)e^{-h/k} \tag{1.25}$$

For the Earth's atmosphere, the factor k is about 7000 m. This means that the pressure decreases by a factor of e every 7000 m, or by a factor of 2 every 5000 m. Even though our atmosphere is not isothermal, the result is not too bad and useful for quick estimates.

The gravitational potential. Pressure differences are interpreted as hydraulic driving forces, pressures are hydraulic "levels." For fluids stacked in the gravitational field, vertical pressure differences are the result of gravity. They are calculated according to Equ. 1.23 or Equ. 1.24. If we multiply the pressure difference by the volume of a certain quantity of liquid which we imagine to be transported from a height h_1 to a height h_2 (Fig. 1.36), we have $\Delta pV = \rho g \Delta h V = g \Delta h \rho V = g \Delta h m$:

$$\Delta pV = -(g\Delta h)m \tag{1.26}$$

This result is interpreted as follows (Fig. 1.36). If we look at gravitational processes as the transfer of the mass of a substance from a level 1 to a level 2, the right hand side of Equ. 1.26 represents mass m going from a gravitational level gh_1 to a level gh_2. Therefore, $g\Delta h$ is interpreted as the *gravitational driving force*, and gh is the so-called gravitational potential.

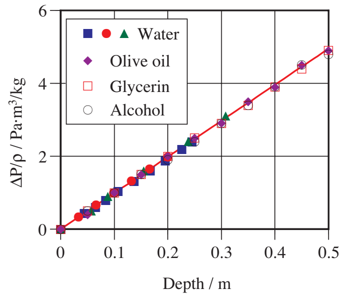

Figure 1.35: Pressure increases downward in a liquid at a fixed "rate" (gradient). The gradient depends upon density and gravity.

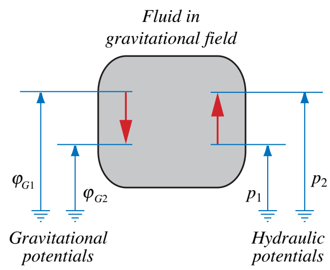

Figure 1.36: A process diagram explaining the coupling of gravitational and hydraulic processes. As a body of mass m goes "downhill by a distance" $g\Delta h$, the pressure of the corresponding volume goes up by Δp.

Questions

60. What is the pressure difference of water at rest inside a very long pipe (10 km) between the two ends. One of the ends is 10 m higher than the other end.
61. The pressure of the air in the atmosphere 5 km up is one-half that at the surface. How much higher do you have to go for the pressure to decrease by another factor of two?
62. What is the meaning of gravitational potential?

1.7 DYNAMICAL MODELS AND SYSTEM BEHAVIOR

Causal physical models are answers to the question "why:" Why is a system in a certain state? Why do processes run a certain way? A complete model of systems and processes is simply a combination of all relations—laws of balance and constitutive laws we have collected so far—necessary for a particular example. The purpose of a model is to determine quantities describing a situation at a moment, or to predict the outcome of processes.

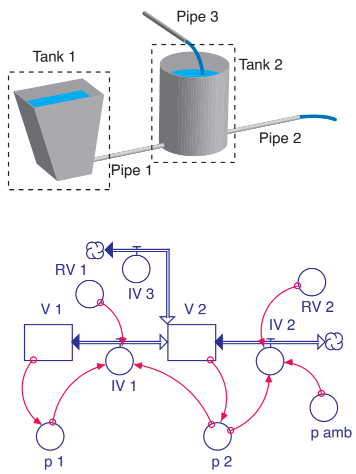

Figure 1.37: A system dynamics model diagram for a system of two fluid tanks (stocks) and several pipes (flows). It is assumed that the fluid is oil and that the Bernoulli effect has been neglected.

1.7.1 Dynamical models

Dynamical models combine laws of balance with the appropriate constitutive laws. They are created by a combination of steps described above in Section 1.3 (Systems analysis I: Laws of balance) and Section 1.4 (Systems analysis II: Pressure and pressure differences), with the particular laws for special processes found in Section 1.6 (Constitutive laws).

If the model describes a dynamical situation, it may be expressed with the help of a system dynamics tool. A system dynamics diagram represents the necessary laws of balance and constitutive laws (Fig. 1.37). Laws of balance are "drawn" graphically by combinations of stocks and flows.

1.7.2 Analytical solutions

Systems made up of containers and pipes show relatively simple behavior. Complex behavior is commonly the result of the interaction of several simple elements. For the simplest systems—those having constant values of capacitance and resistance—analytic solutions of the model equations can be obtained. In the case of draining straight-walled tanks through horizontal pipes with laminar flow we get

$$\Delta p(t) = \Delta p_o e^{-\frac{t}{R_V C_V}} \tag{1.27}$$

If an empty tank is charged, the solution of the model is

$$\Delta p(t) = \Delta p_{max}\left(1 - e^{-\frac{t}{R_V C_V}}\right) \tag{1.28}$$

We have observed this function before. The formal solutions found here correspond to the curves seen in Fig. 1.13. Equ. 1.27 and Equ. 1.28 also hold for $h(t)$ and $V(t)$.

1.7.3 Time constants

The behavior (fluid level as a function of time) for the simple cases of draining and filling of a tanks is shown in the accompanying graphs (Fig. 1.38). The solutions of the model are exponential functions. A measure of how fast (or slow) the process is, is the time it would take for the tank to drain or to fill were the level to continue to change at the initial rate. This time is called the capacitive time constant τ_C of the system. In one time constant, the level of fluid in the system shown on the left in Fig. 1.38 drops to $1/e$ = 0.37 times the initial level. The analytic solutions in Equ. 1.27 and Equ. 1.28 demonstrate that

$$\tau_C = R_V C_V \tag{1.29}$$

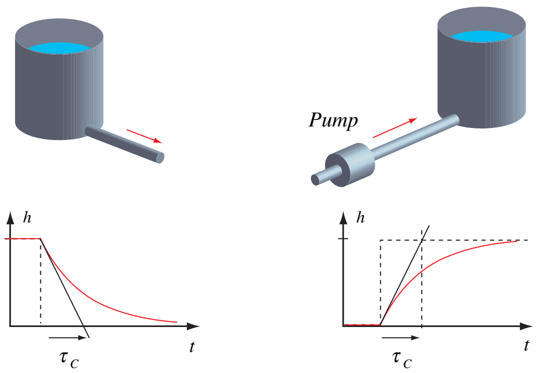

Figure 1.38: Draining or filling of straight walled tanks through pipes showing laminar flow leads to exponentially changing functions. The initial rate of change is used to define the time constant of the system.

1.7.4 A windkessel model

A windkessel – pump system (Fig. 1.39, left) is basically a container with an input driven by a pump and an outflow through a hose. If the pump works intermittently, i.e., if the driving force is variable, the output is smoothed or dampened. Its variability is reduced relative to the variability of the input. There is a valve between the pump and the container to prevent the liquid to flow back into the pump. Such a windkessel-model is used to represent the functioning of heat and aorta. Historically, windkessel pumps were developed for fire fighting.

In this activity we will use a dynamical model to investigate the behavior of the system (Fig. 1.39) if the pressure of the pump is constant during a short period of a cycle, and zero for the rest of the cycle. This is repeated once every cycle. In particular, we want to find out about the pressure of the liquid in the container.

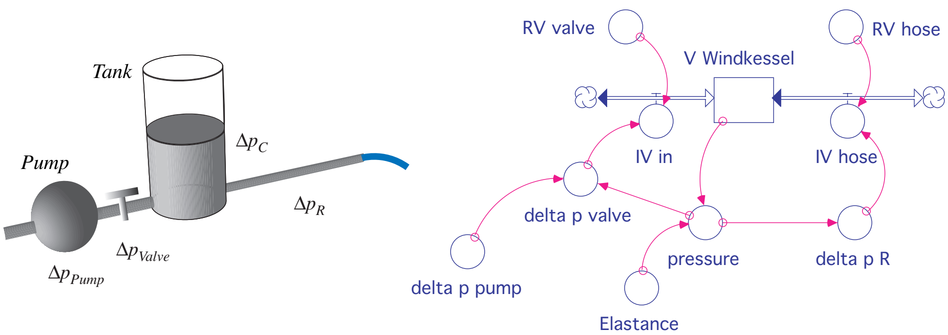

Figure 1.39: Windkessel model with pressure differences (left). System dynamics diagram of model (center) and equations (right). Note that the driving pressure of the pump has been expressed by a square wave function to obtain the result shown in Fig. 1.40.

A system dynamics model starts with a representation of the law of balance of volume for the tank: a stock with an inflow and an outflow. The flows depend upon the associated pressure differences, whereas the pressure of the liquid at the bottom of

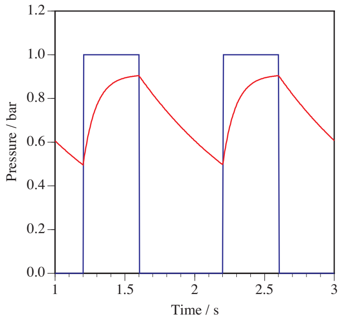

Figure 1.40: Response of the windkessel (oscillating pressure of the fluid in the tank with values between 0.5 bar and 0.9 bar) to the forcing function of the pump (square wave).

the tank is calculated with the help of the capacitive relation of the container. The pump is modeled by the pressure it sets up as a function of time (Fig. 1.40), and the valve is represented by an if…then…else construct for the flow from the pump to the tank.

The response of the system to the action of the pump is demonstrated by the pressure of the liquid in the tank, here shown as the curve in Fig. 1.40 that oscillates between a high and a lower value (between 0.9 bar and 0.5 bar). This, by the way, explains the phenomenon of blood pressure measured in the aorta of a person or an animal which changes between systolic and diastolic values.

Note that, for times when the pump is turned off, the pressure function is a decaying exponential (the flows in the model have been taken to be laminar). If the pump remained turned off after some time, the tank would drain according to the rules we have seen at work before (Fig. 1.38). We can use Fig. 1.40 to determine the time constant of the system (made up of the tank and the long pipe leading away from it) which is 1.0 s. Knowing that the time constant must be equal to the product of resistance and capacitance, one of these values may be obtained if the other one is known.

QUESTIONS

63. In what sense can we say that a model such as the one in Fig. 1.37 explains a system?
64. Explain the meaning of the structure of stocks (rectangles) and flows (thick arrows) in the system dynamics diagram of Fig. 1.37. Why are there three flows connected to the stock V2?
65. In Fig. 1.37, why are p1 and p2 connected to the quantity labeled IV1?
66. To what percentage of the final level does the level on the right in the diagram of Fig. 1.38 rise in one time constant?
67. Consider a system of two communicating tanks with oil as in Fig. 1.1. How can you use the diagram there to graphically determine the time constant of the system? What is your estimate of this time constant?