CHAPTER 6 Rotational Mechanics

In this chapter, simple single-dimensional rotational processes will be treated. Essentially, we will be concerned with wheels rotating around fixed axes. It will become obvious that rotational motion can be understood in analogy to fluid or electric processes.

6.1 Some Important Observations

Establishing rotational equilibrium. Imagine two rotating bodies sharing a single fixed axis. This could be a person standing on a turning table holding a bicycle wheel above her head (Fig. 6.1). The wheel is turning, the person stands still. Now she touches the tire of the wheel with her hand, slowly braking it. As a result, the person will start turning while the rotational speed of the wheel decreases. When the two speeds have become equal, i.e., when the hand has the same speed as the tire and there is no more friction, the process of adjustment of speeds stops. If the turning table had perfect roller bearings, and if the air did not brake the rotating system, the person and the wheel would go on turning forever at the same angular speed.

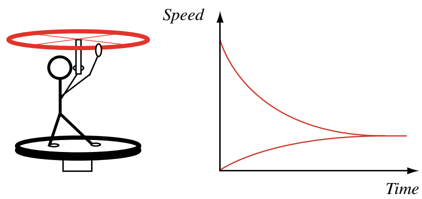


Figure 6.1: A person on a turntable holds a spinning bicycle wheel and brakes it with her hand. The resulting rotational speeds of the wheel and the person are recorded.

Interpretation. What we see here looks suspiciously like what we have seen in the case of communicating water containers, or communicating capacitors. We can understand what we see by assuming that a rotating body has a *quantity of rotational motion* it can pass on to other bodies. Here, this quantity is passed from the tire to the person as a result of the difference in angular speeds of hand and tire. When this difference has become zero, the transfer of quantity of rotational motion stops.

Rotational collisions. Two wheels spin around the same vertical axis as in Fig. 6.2. The upper one is made to spin, the it is lowered onto the stationary lower one. The wheels couple and spin together (they collide). Depending on the relative size of the two wheels, the final rotational speed is in a special relation with the initial speed of the upper wheel. The “size” of the wheel depends upon its mass and its radius; if either one of these factors is changed, the rotational properties of the wheel change.

For example, if the wheels are identical, the final angular speed of the two wheels together is half that of the upper wheel spinning by itself. If the upper wheel has the same radius but half the thickness (i.e., half the mass), the final angular speed is two thirds of the initial speed. The same result is obtained when the upper wheel has the same mass but only 71% of the radius.

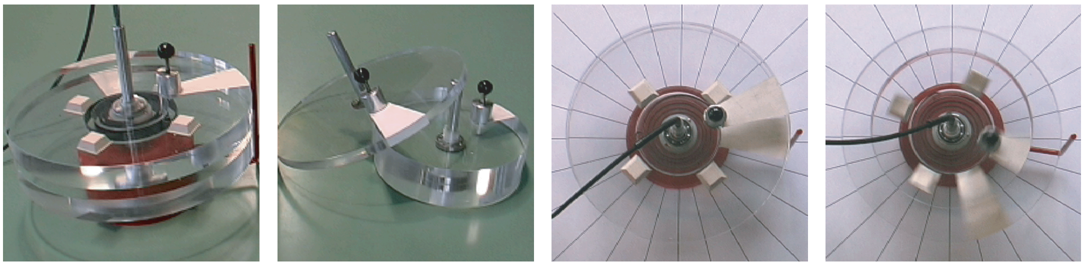


Figure 6.2: Two wheels can spin on the same vertical axis, either separately, or coupled together. In the third picture, two identical wheels are mounted. In the fourth picture, the upper wheel has the same mass as the lower one, but only 71% of its radius.

Interpretation. We can use the same basic idea applied before. The spinning wheel has a certain quantity of rotational motion (there are different words for this concept; one of them is *spin*). In the case of coupling with an identical wheel, half of the spin should be communicated to the lower wheel. Each wheel is observed to spin half as fast as the first wheel. This means that the quantity of rotational motion is proportional to the angular speed.

The observations with unequal wheels tells us something about the relation between spin and angular speed for a particular wheel; in analogy to hydraulics or electricity we would speak of the capacitance for quantity of angular motion. A wheel with half the mass but the same radius has half the capacitance. A wheel with the same mass but only 71% of the radius also has half the capacitance. This means that the capacitance for spin is proportional to the mass of the wheel, and to the square of the radius (71% is $1/\sqrt{2}$).

Driving wheels apart with springs. The experiment can be used to demonstrate another interesting phenomenon. The wheels in Fig. 6.2 can be joined by a twisted spring (Fig. 6.3). If we let go of the wheels, they start rotating in opposite directions. Depending on the mass and the radius of the wheels, the finals angular speeds of the wheels stand in a particular relation. If the wheels are identical, the speeds have the same magnitude but opposite sign.

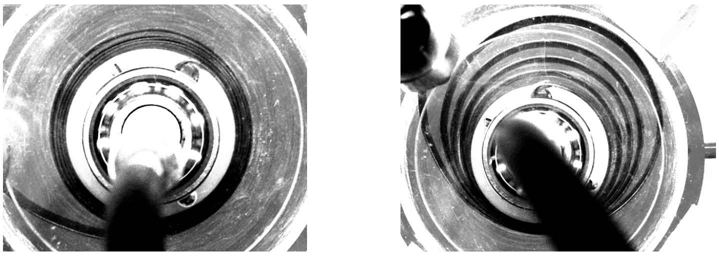


Figure 6.3: Twisted torsion spring. Such springs, when placed between two wheels, can be used to drive the wheels apart. (Left: Twisted spring; right: relaxed spring.)

Interpretation. As in electricity, we say that the fundamental quantity used to explain the phenomena can take positive and negative values (charge in electricity, quantity of rotational motion or spin in rotation). If the wheels were at rest initially, the wheels will acquire equal but opposite amounts of rotational motion. The torsion spring "pumps" spin from one wheel into the other. Overall, the quantity of rotational motion remains the same.

Friction in rotational motion. In the first example, friction between the hand and the tire of the bicycle wheel makes the wheel slow down. In the second example, a wheel spinning around the vertical axis will eventually come to a halt (Fig. 6.4). Also, the real values reported for the final angular speeds in the rotational collisions are smaller than the values reported above.

Interpretation. All of these phenomena are explained by assuming that a spinning wheel looses spin because of friction. The quantity of rotational motion is transferred to the body joining in the process of friction.

Torsion. Take a closer look at some processes of friction in rotational motion. A wheel rotating about an axis and slowing down interacts with the axis. To get a clear feeling for this, take a stick, put your left hand around it, and use the other hand to turn the stick inside the left hand. There is strong friction between the stick and the palm of you hand, and you can notice what the stick is doing to your hand: It twists the skin in the direction of rotation. The stick itself, if it were somewhat flexible, would clearly be twisted as well in the opposite direction.

You can observe this twisting of a body also in the case of the person with bicycle wheel on a turntable. As soon as we touch the spinning wheel, our hand—and with it the arm and the body—is pulled in the direction of motion of the tire. As a result, our body twists in the direction of rotation of the wheel.

This phenomenon—called torsion— is also observed if nothing rotates. In the case of the stick in your left hand, if you hold it strongly enough while twisting it with your other hand, you notice the same phenomenon: Stick and hand are twisted.

Interpretation. In the cases of bodies moving relative to each other, we say that quantity of rotational motion (spin) is transferred from one of the bodies to the other. We associate this with a difference in angular speed: We can say, that quantity of rotational motion flows from a point of high angular speed to one of lower angular speed (Fig. 6.5). The flow of spin is normally called *torque*.

Since torsion also occurs without motion, we should conclude that spin is transferred through the twisted material in this situation as well. Here we have an interesting new phenomenon: Quantity of rotational motion flows through materials even if they do not move. It is enough that they are twisted or bent. Since there is no difference of angular speed associated with the process, we have a transport that does not need a "driving force." Such phenomena are actually well known in electricity. It is possible to let electricity flow through superconductors without needing a voltage. Therefore, we shall speak of the transport of spin without a difference of speeds as a "superconducting" transport.

Clearly, the stronger the effect of torsion, the stronger the flow of spin (torque). A wheel will spin down (lose spin) faster if torsion is stronger (the curve in Fig. 6.4 will decay faster).

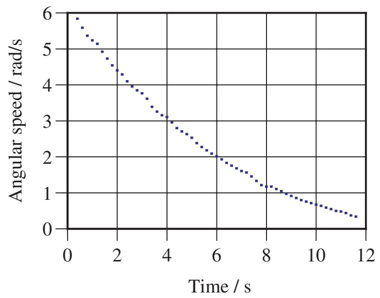


Figure 6.4: Angular speed of a decelerating spinning wheel. Notice the similarity with phenomena in other fields (discharging of a tank, cooling of hot water, etc.).

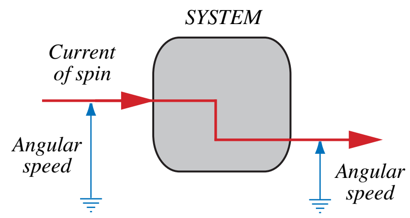
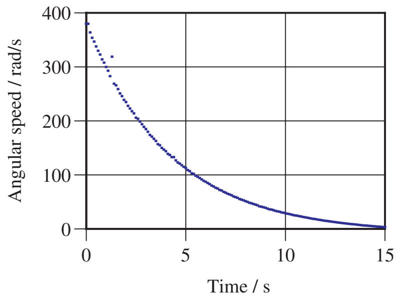


Figure 6.5: Process diagram of a typical rotational process. Quantity of rotational motion flows from points of high to points of low angular speed.

Braking a wheel electrically. A flywheel is attached to an electric generator. It spins and drives the generator. If we hook up a resistor to the generator, the wheel will spin down as shown in Fig. 6.6. As expected, the resistor will get warm. The measured curve is close to an exponentially decaying function.

Interpretation. The wheel spins down which means that it loses spin (quantity of rotational motion). This spin is flowing through the axel that drives the generator where it "falls" to a lower level, meaning to an angular speed of zero (the angular speed of the ground, Fig. 6.6).



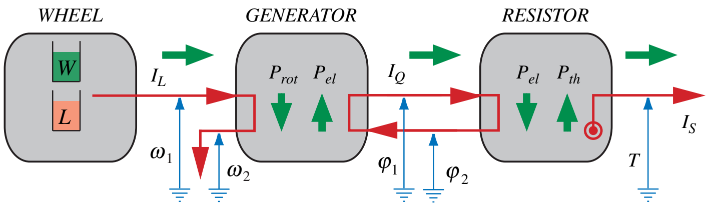


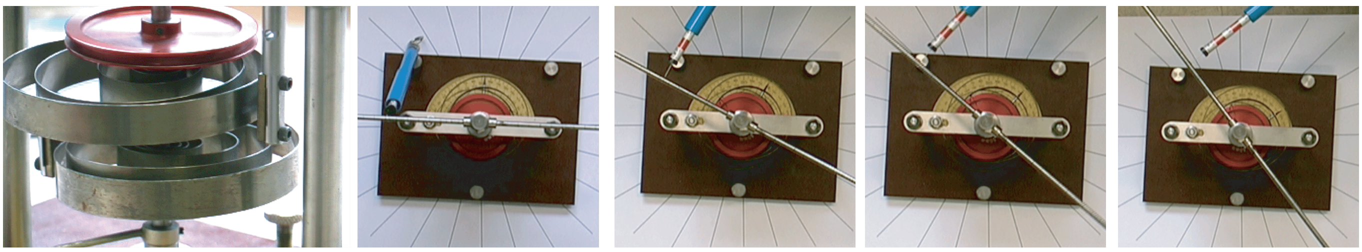
Figure 6.6: A flywheel drives a generator which is hooked up to a resistive lead. The angular speed decreases (top). The process diagram of wheel, generator, and lad is shown on the left. L is the symbol for spin, ω stands for angular speed.

As the quantity of rotational motion "falls" to a lower level, energy is released. This energy was first transferred from the wheel to the generator. The energy released in the generator is used to drive the electric process which, in turn, drives the thermal process of entropy production in the load resistor. In summary, we interpret the wheel as an energy storage device, and spin as the energy carrier from the wheel to the generator. The role of energy in rotational processes appears to be the same as in all other processes studied so far.

The angular speed should be an exponentially decaying function of time in a model with only linear characteristics.

Torsion springs. Thin metal strips can be wound to torsion springs. These springs have simple properties: The strength of the current of spin (torque) passed by a twisted torsion spring depends linearly upon the twist angle: Double the twist angle yields double the torque.

Figure 6.7: Twisted spring. Such springs pass a current of spin that depends upon the twist angle. The torque can be measured by the product of the force of pull and the distance of the point where we pull from the axis of rotation.



Measurement. This relation can be determined in at least a couple of different ways. We may use dynamometers ("Newton-meters") and pull in a perpendicular direction on a rod attached to the spring (Fig. 6.7). The force read from the dynamometer, multiplied by the distance of the point where we pull from the axis of rotation, gives the torque. This determination leads to the same values as in the following procedure. We can attach a wheel to the axis, twist the spring, and let the wheel accelerate. From this we determine the flow of spin (torque).

There is another way at looking at a torsion spring. Dynamically speaking, a torsion spring relates how fast the twist angel changes to how fast the torque (current of spin) changes. Since the rate of change of the twist angle equals the angular speed, and since the angular speed is the rotational potential, this statement is analogous to the law of induction. Therefore, a torsion spring is a rotational *inductive* element. Just like any other inductor, a spring can store energy as it passes a current of spin.

Rotational oscillations. If a body is attached to an axle driven by a torsion spring, the body will perform rotational oscillations (Fig. 6.8).

Interpretation. The most direct way of seeing why this system is an oscillator is to compare it to hydraulic or electric oscillators. These consist of capacitive and inductive elements (usually in conjunction with resistors). The rotating body is a capacitor (it contains spin when it rotates), the spring is an inductive element, and there typically are resistive factors as well. Seen from the perspective of energy, we charge the spring with energy when we rotate the body out of its position of equilibrium. This energy is first passed to the rotator, then back to the spring, and so forth. If there is friction (resistive elements), the oscillation is damped (Fig. 6.8). As in the case of hydraulic or electric oscillators, the (eigen)frequency of the system will depend upon the rotational capacitance of the rotator and the inductance of the torsion spring.

Resonance. If a an oscillator is driven harmonically (with a sinusoidal torque, see Fig. 6.9, left), it will exhibit resonance. For a fixed frequency of the drive, the motion will finally reach a steady oscillation. The amplitude of this oscillation depends upon the drive frequency, and it will reach a maximum for a drive frequency close to the eigenfrequency of the oscillator (see the graph on the right in Fig. 6.9; there are resonance curves for different values of damping).

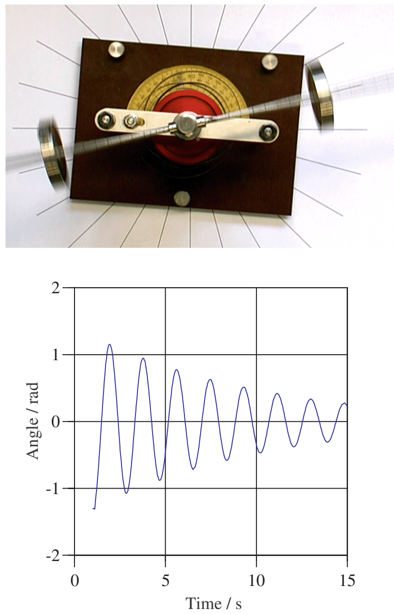


Figure 6.8: A rotating body attached to an axle with a torsion spring. The body performs rotational oscillations around the axle.

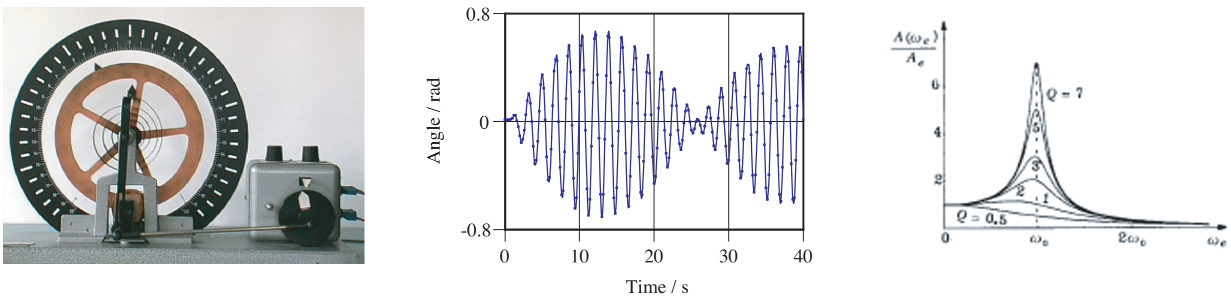


Figure 6.9: A torsional oscillator (such as Pohl's pendulum consisting of a copper wheel, a torsion spring, and an eddy current brake) can be driven mechanically. A motor and a simple device change the twist angle sinusoidally. This leads to a harmonically driven oscillation which exhibits resonance. (right).

6.2 Angular Momentum and Angular Speed

The simplest situation of a rotating body is that of a rigid body spinning around a fixed axis. Rotating bodies have *angular momentum* or *spin* (the faster they spin, the more they have), a quantity that fits our everyday notion of momentum, simply transferred to rotation. We think of angular momentum to be stored in (rotating) bodies, and to be able to be added to or taken away from these bodies (it can flow). Experience leads us to conclude that it cannot be produced or destroyed.

6.2.1 Processes and angular momentum transfer

A body spinning freely (without any interference of the outside world) would continue to spin forever. Rotational processes, therefore, a conceptualized as consisting of the transfer of angular momentum.

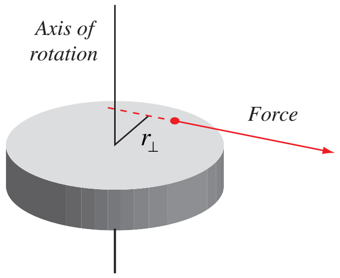


Figure 6.10: Relation between a single force and the associated angular momentum current.

Different interactions between a body under consideration and its environment lead to different forms of angular momentum transfer. The stronger the interaction, the stronger the *flow of angular momentum* (I_L). Angular momentum currents are called *torques* in mechanics.

Imagine a wheel spinning around a fixed axis. Angular momentum transfer can be the result of someone or something pulling or pushing on the periphery, friction between the wheel and the axis, air resistance, magnetic fields interacting with metallic discs (eddy current brakes), the action of torsion springs, etc. Torsion and bending are the typical responses of materials to the transfer of angular momentum.

Angular momentum transfers can be intrinsic, or they can be associated with a force acting on a body (see Chapter 7). In the latter case, it matters how far from the axis the force is acting. Here, the torque is related to the distance and the force Fig. 6.10):

$$I_L = r_\perp F \tag{6.1}$$

6.2.2 Balance of angular momentum

The assumed properties of angular momentum let us postulate a *law of balance of angular momentum*:

$$\dot{L} = I_{L,net} \tag{6.2}$$

In integrated form, it is

$$\Delta L = L_{e1} + L_{e2} + \ldots \tag{6.3}$$

6.2.3 Dependence of angular momentum upon angular speed

Intuitively, it is clear that the angular momentum (spin) of a body depends upon its rotational speed (*angular velocity*; abbreviated by ω). Higher angular speed means more spin. Simple observations with interacting wheels (Fig. 6.2) indicate that angular momentum and angular velocity are proportional:

$$L = J\omega \tag{6.4}$$

The factor of proportionality tells us how much angular momentum a body need to spin at a certain rate. The bigger and the heavier, the larger J. Therefore, this factor measures a kind of inertia of the spinning body. It is called the *moment of inertia*. Comparison to electricity or hydraulics tells us that the moment of inertia is the *angular momentum capacitance* of a body.

Observations such as in Fig. 6.1, and the form of the relation in Equ. 6.4, suggest that the angular speed serves the role of the rotational potential (rotational level). The relation between spin, angular speed, and moment of inertia can be symbolized by the fluid image in Equ. 6.11.

For bodies whose moment of inertia is constant, we can rewrite the law of balance of angular momentum. In this form it is called Euler's law:

$$J\dot{\omega} = I_{L,net} \tag{6.5}$$

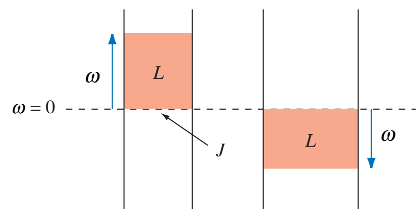


Figure 6.11: Fluid image of angular momentum. Angular momentum and angular speed can be positive or negative, just as electric charge. The cross section of the symbolic storage tank for angular momentum corresponds to the moment of inertia of the body.

6.3 Rotational Kinematics

Kinematics is about three geometric concepts and their relations: angular displacement (angle), angular velocity, and angular acceleration.

6.3.1 Angular speed and angle

The angular velocity tells us how fast the angle (angular orientation) of a body changes. Therefore, angular speed is the rate of change of angle (Fig. 6.12):

$$\omega = \dot{\phi} \quad , \quad \phi(t) = \phi_0 + \int_0^t \omega(t)\,dt \tag{6.6}$$

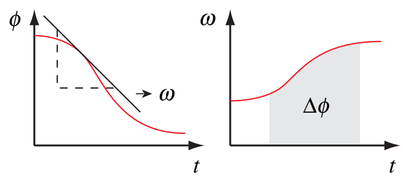


Figure 6.12: Relations between angle (angular displacements) and angular velocity.

6.3.2 Angular speed and angular acceleration

The angular acceleration tells us how fast the angular speed of a body changes. Therefore, angular acceleration is the rate of change of angular velocity (Fig. 6.13):

$$\alpha = \dot{\omega} \quad , \quad \omega(t) = \omega_0 + \int_0^t \alpha(t)\,dt \tag{6.7}$$

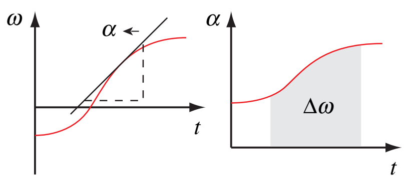


Figure 6.13: Relations between angular velocity and angular acceleration.

6.4 Energy in Rotational Motion

Energy takes the same role in rotational phenomena as in all other types of physical processes studies so far.

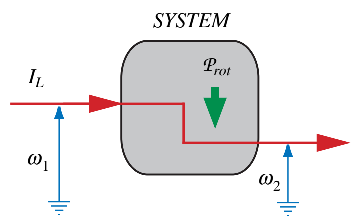


Figure 6.14: Power in a rotational process: Energy is released at a certain rate when angular momentum flows from points of high to points of low angular speed.

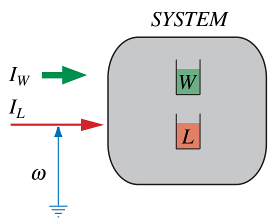


Figure 6.15: Energy transfer together with angular momentum depends upon the particular angular speed at which angular momentum is transferred.

6.4.1 Rotational power

When angular momentum flows from a point of higher to a point of lower rotational potential (i.e., angular speed), energy is released (Fig. 6.14). In the reverse process, energy is bound. The rate at which energy is released is called *rotational power*:

$$\mathcal{P}_{rot} = \Delta\omega I_L \tag{6.8}$$

6.4.2 Energy transfer

When bodies interact with their environment, angular momentum is transferred. If the body rotates, the angular momentum flow (torque) is accompanied by an energy current:

$$I_W = \omega I_L \tag{6.9}$$

6.4.3 Energy storage

The relation between energy stored and relevant factors depends upon the type of storage device. It has to be derived from our knowledge of particular circumstances.

Energy can be stored in rotating bodies. The faster a body spins, and the greater its moment of inertia, the more energy it contains because of rotation:

$$W = \frac{1}{2}J\omega^2 \tag{6.10}$$

Equally, energy can be stored in twisted torsion springs:

$$W = \frac{1}{2}D'(\Delta\phi)^2 \tag{6.11}$$

D' is called the torsion spring constant, $\Delta\phi$ is the twist angle. By the way, torsion springs are rotational inductive elements (Section 6.6.2).

6.5 MOMENT OF INERTIA

Observations such as in Fig. 6.2 indicate that the moment of inertia of a rotating body depends upon its mass and the distribution around (distance from) the axis of rotation. It is important to note that the moment of inertia of a given body changes if the axis of rotation is shifted.

6.5.1 Moment of inertia for center of mass axis

Our observations show that the moment of inertia of a small body is proportional to its mass and to its distance squared from the axis of rotation. In this expression, the size of the body should be very small compared to its distance from the axis:

$$\Delta J = r^2 \Delta m \tag{6.12}$$

The moment of inertia of an extended body is found by dividing it into small parts and summing the results of Equ. 6.12 over all parts. This is the integral

$$J = \int_m r^2 dm \tag{6.13}$$

Examples of moments of inertia calculated for different bodies with given axis of rotation are summarized in Table 6.1. Examples are all for axes going through the center of mass of a given body.

Table 6.1: Moment of inertia of some bodies[a]

Body and axis	Moment of inertia
Thin ring of mass M having a radius R. Axis of rotation through center, perpendicular to plane of ring.	MR^2
Homogeneous disk of mass M and radius R. Axis through center, perpendicular to disk.	$\frac{1}{2}MR^2$
Ring of mass M having inner and outer radii R_1 and R_2. Axis of rotation through center, perpendicular to plane of ring.	$\frac{1}{2}\left(R_1^2 + R_2^2\right)M$
Thin rod of mass M and length l. Axis through center, perpendicular to axis of rod.	$\frac{1}{12}Ml^2$
Homogeneous sphere of mass M and radius R. Axis through center.	$\frac{2}{5}MR^2$

a. Expressions for more complex bodies can be obtained by combining bodies from simple parts, and by using Equ. 6.14.

6.5.2 Shifting the axis of rotation

If the axis of rotation is different from the axis for which an expression for the moment of inertia was obtained, the result changes. Given the moment of inertia for a center of mass axis J_{CM}, it can be calculated for another axis parallel to this center of mass axis:

$$J' = J_{CM} + s^2 M \tag{6.14}$$

s is the distance by which the axis was shifted, and M is the mass of the body.

6.6 SOME CONSTITUTIVE LAWS

Models of dynamical rotational processes can only be constructed if the special laws for torques (angular momentum flows) pertaining to particular situations are known.

6.6.1 Friction

Constant friction. The angular momentum flow (torque) is constant, but its sign depends upon the sign of the angular speed (i.e., upon the direction of rotation):

$$I_L = -\text{sign}(\omega)c \tag{6.15}$$

Ohmic type friction. The magnitude of a friction effect may depend upon the relative rotational speed between the two bodies interacting. If there is an oil film between the surfaces of the bodies, the effect is typically proportional to the angular speed difference:

$$I_L = -k\omega \tag{6.16}$$

This linear characteristic is analogous to Ohm's law in electricity.

Eddy current brakes. Rotating conducting metal wheels can be braked with the help of magnets. The effect of a magnetic field perpendicular to a rotating conducting disk is proportional to the strength of the magnetic field and to the angular speed of the disk. Therefore, the torque law is linear.

6.6.2 Torsion springs

Torsion springs can be made elastic over a large twist angle. Moreover, their effect upon a body they are attached to, i.e., their torque, is usually linearly dependent upon the twist angle $\Delta\phi$:

$$I_L = -D'\Delta\phi \tag{6.17}$$

D' is called the torsion spring constant. By calculating the rate of change of the angular momentum current, we see that

$$\frac{1}{D'}\frac{dI_L}{dt} = \omega \tag{6.18}$$

Since ω is the rotational potential, this equation is of the form of a law of induction. Torsion springs are rotational inductive elements. Their *inductance* is the inverse of the spring constant.

6.7 DYNAMICAL MODELS OF ROTATIONAL SYSTEMS

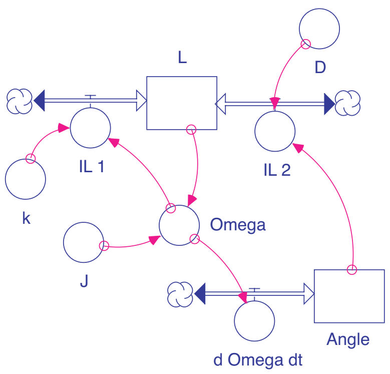


Figure 6.16: System dynamics diagram of a typical dynamical model of a rotating body. Note the law of balance of angular momentum L, the expression for angular speed Omega, the integration of omega to yield the angle, and the expressions for the angular momentum flows.

There is a strong similarity between hydraulic, electric, thermal, and rotational systems. We can build dynamical rotational models as we have done for hydraulic or other systems. Starting point is the law of balance of angular momentum of a rotating body (Fig. 6.16). The angular speed is calculated from the angular momentum (with the help of the moment of inertia).

There is a new element, however: Kinematics tells us that the angle of rotation can be calculated from the angular speed by simple integration (a combination of stock and flow in standard system dynamics models, Fig. 6.16). The expressions for the angular momentum flows can then be determined from angle, angular speed, and other factors.

System behavior. A simple system might be made up of a rotating disk and an eddy current brake. Analogical reasoning tells us that the disk should spin down exponentially which, in fact, it does (Fig. 6.6). For such a system we can introduce the *time constant* $\tau = J/k$. This is analogous to the expression $\tau = RC$ for an electrical *RC* system.