CHAPTER 1

The Storage and Flow of Fluids

CONTENTS

Water is stored in artificial lakes, and it flows to drive other processes.

The flow of water at the surface of the Earth serves as the prime source of our understanding of physical processes. We can see nature at work in waterfalls, rivers, and lakes. Technical examples—the water supply of homes, heating oil storage, hydraulic power plants, or hydraulic control systems—teach us about the physical quantities and relationships necessary for a description of hydraulic phenomena.

In the first section of this chapter we will construct the model of the dynamical processes occurring in a system of two communicating lakes. Without going into the underlying theory, we will simply create a model and simulate it. We will see in this section how we can transform our fundamental ideas about how nature works into quantitative models which allow us to actually calculate the outcome of processes.

1.1 A Simple Model of Fluid Storage and Flow

Imagine two small lakes fed by a single stream. The lakes communicate through an underground channel. The second lake drains through another underground passage into the open (Fig.1.1). It is known that, during a long period when the stream feeding the first lake carries a steady current of water of 10 liters per second, the levels of water measured from the bottom are 10 m and 8 m, respectively. The surface areas of the lakes are 200 m^2 and 400 m^2.

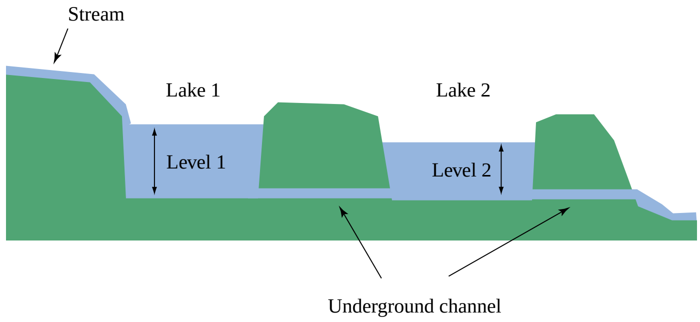


FIGURE 1.1. Two lakes are communicating through an underground channel. The first lake is fed by a stream, the second drains through another underground channel.

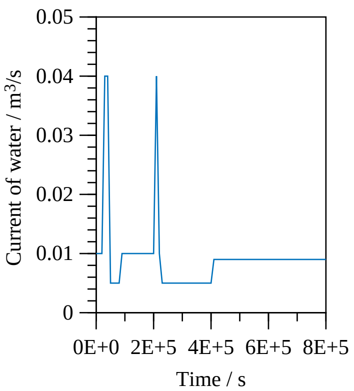


FIGURE 1.2. The current of water feeding the first lake as a function of time. We wish to construct a model of the lakes which allows us to calculate how the system reacts to the changes of the current of water feeding them.

Rain lets the stream feeding the lakes swell. After each rain fall, the current of water is found to decrease to 5 liters/s for s short period. It increases again to a steady value of 10 liters/s after the first rain, and to 9 liters/s after the second rain (Fig.1.2). We wish to construct a model of the flow and the storage of water which allows us to calculate, and therefore predict, the water levels in the lakes as a result of the changes of the current feeding the lakes.

A Partial Model: A Single Lake With a Single Flow of Water

We will approach this first example of a model of a dynamical system by constructing it in small steps. First, assume that we only have one lake which is fed by a stream without being drained. In other words, we have a single container for water with a single flow. Obviously, when constructing a model of this situation, we must create a descrip-

Cover picture: Lake Cleuson near Nendaz in the Swiss Alps; Grand Coulee Dam.

tion which uses containers and flows to mimic what we see in reality. Our first step consists of drawing a flow connected to a container (Fig.1.3).[1] This diagram represents our belief that the flow determines how the content of the container will change. In other words, it represents the *law of balance of amounts of water* necessary to compute what is happening in the system.

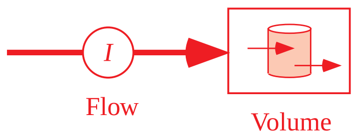


FIGURE 1.3. System dynamics diagrams of dynamical processes start with stocks and flows, representing system contents and currents of substancelike quantities (such as the volume of water).

We have to introduce abstract quantities which let us specify numerical values associated with how much water is in the lake, and how large the current of water is in the course of time. For the former we simply use the *volume of water* (measured in cubic meters), and for the latter we take what is called a current or a *flux of water* (measured in cubic meters per second). In Fig.1.3, the flow is drawn as a fat arrow with a circle containing the symbol I_X which we will use to denote currents or fluxes, while the volume of water contained in the lake at any moment is represented by a box which is called a stock. Stocks and flows make up the backbones of the models we are going to construct in this book.

Now, we have to think carefully about the form of laws of balance such as the one we just formulated graphically. So far we said that—if there is only one flow—the flux determines how the amount of water in the container is going to change. In fact, the volume flux of water determines *how fast* the volume must change. The formal quantity measuring how fast a system content changes is called the *rate of change* of this quantity, and it is denoted by the symbol for the content with a dot written above it. Thus, the rate of change of the volume V of water in the system is written as $\dot{V}$. If we write I_V for the flux of volume associated with the single flow of water, the law of balance of the volume of water of the lake is written in the following form:

$$\dot{V} = I_V \tag{1.1}$$

In other words, whenever we express our ideas of the storage and the flow of a substancelike quantity in the form of a diagram containing stocks and flows as in Fig.1.3, we can immediately write down the law of balance in mathematical form.

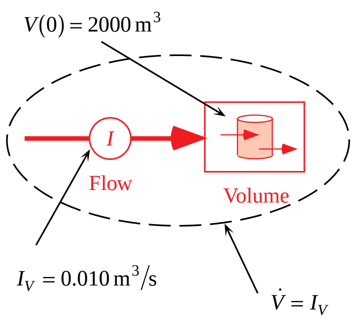


FIGURE 1.4. A combination of a stock and flow(s) represents a law of balance. Here, this law relates the flux of water to the rate of change of the water content. The flow of water and the initial value of the content must be specified for a solution of the law if balance to be possible.

Calculating the volume of water in the lake as a function of time calls for solving the law of balance expressed in Eq.(1.1). By the way, this step is called simulating the model. However, before we can do this for our miniature model of a lake with one stream feeding it, we need to know two things: first, the flux I_V must be know for the period of time for which we wish to compute the content of the lake, and we have to specify the amount of water in the lake at the beginning of this period. The latter quantity is called the *initial value* of the volume V. For our example we will choose a constant flow of water of 10 liters/s. Therefore, the complete set of relations specifying our model (Fig.1.4) in mathematical terms is

$$\begin{aligned} \dot{V} &= I_V \\ V(0) &= 2000\,\mathrm{m}^3 \\ I_V &= 0.010\,\mathrm{m}^3/\mathrm{s} \end{aligned} \tag{1.2}$$

A Simple Numerical Method for Solving the Equations

We have to think of a method of solving the set of equations. First, we have to note that the solution does not consist of just three numbers, but of an infinite number of values

1. Here, we adapt the diagramming technique known from system dynamics tools such as Stella to our needs in physics, which are almost identical to the standard system dynamics terminology.

of the volume at all points in time in the interval of interest. In other words, the solution of Eqs.(1.2) is the function $V(t)$. Now, solving Eqs.(1.2) is not all that difficult; we know that the volume of water in the lake changes as the result of the influx of water. Knowing the flux lets us find out how much water has been flowing into the lake during a small period of time which we call a *time step* Δt, namely $I_V \Delta t$. The amount of water added to the lake (or withdrawn from it) during a specified period of time is called the amount *exchanged*, and it is abbreviated by V_e:

$$V_e = I_V \Delta t \tag{1.3}$$

Note that this expression is correct only if the flux of water is constant during the time step considered. If this method of determining the volume exchanged is not accurate enough, we can change either the time step Δt or the method embodied by Eq.(1.3).

If we now add this amount of water communicated to the lake to the amount of water in the lake at the beginning of the time step, we obtain the volume of water at the end of the step:

$$V_{new} = V_{old} + I_V \Delta t \tag{1.4}$$

This is a simple procedure for solving sets of equations of the type found in Eqs.(1.2). It can be implemented in the form of a spread sheet which, in our simple case, can be set up and computed manually (Table 1.1).

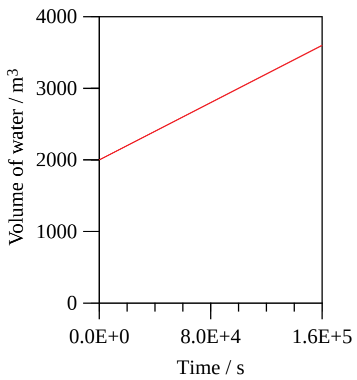


FIGURE 1.5. The solution of a set of equations such as Eqs.(1.2) is given in terms of the volume as a function of time. The result can be presented in form of a graph of V versus t.

TABLE 1.1. **Spread sheet for calculating the solution of Eqs.(1.2).**

Point i	**t_i / s**	**$I_V(t)$ / m^3/s**	**V_e / m^3 from t_i to t_{i+1}**	**$V(t)$ / m^3**
1	0	0.010	400	2000
2	40000	0.010	400	2400
3	80000	0.010	400	2800
4	120000	0.010	400	3200
5	160000	0.010		3600

Typically, we present the solution of the set of equations in the form of a diagram, i.e., we display V graphically as a function of time (Fig.1.5). In our example, the water content of the lake increases linearly with time, which is the result of the single constant flow of water into the system.

Before we continue extending the model of the system we will change the expression for the flow of water feeding the lake. Let the flux of water increase linearly from 10 liters/s at $t = 0$ s to 18 liters/s at $t = 160{,}000$ s. The equations to be solved are

$$\begin{aligned} \dot{V} &= I_V \\ V(0) &= 2000\,\mathrm{m}^3 \\ I_V &= 0.010\,\mathrm{m}^3/\mathrm{s} + \frac{0.001}{20000}\,\mathrm{m}^3/\mathrm{s}^2\, t \end{aligned} \tag{1.5}$$

If we use the same time step of 40,000 seconds as in the previous example, we will get the results presented as the lower curve in Fig.1.6. If, on the other hand, the number of steps is doubled, we obtain a different solution (center curve in Fig.1.6).

Our algorithm for solving the equations (Eq.(1.4)) only yields an approximation to the true solution which improves with decreasing size of the time step. (In our first example, Table 1.1, the result is accurate only because the flow is constant.) If we decrease the time step used for the second calculation even further, the result will be still more accurate. Fig.1.6 compares the numerical solutions with the analytical result calculated with the help of formal rules.

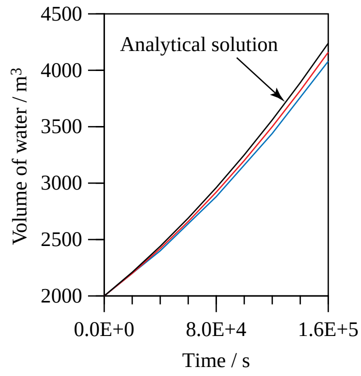


FIGURE 1.6. Comparison of two numerical solutions of Eqs.(1.5) with the theoretical result. The lower curve is the result found in with a time step of 40000 s, whereas the one in the middle is obtained with Δt = 20000 s.

A Lake and Two Flows

Let us extend the model by assuming that the first lake also drains through a second flow. This assumption is easily taken into account by adding a second flow to the stock in the system dynamics diagram of Fig.1.3. In addition, we want to calculate the water level in the lake.

The second flow is again drawn as an arrow pointing toward the stock, even though it is supposed to represent a stream of water flowing out of the system. However, we can deal with this situation by giving the flux a negative value. Assume that the inflow is constant an equal to 0.010 m^3/s, while the outflow increases linearly from – 0.015 m^3/s to – 0.007 m^3/s in the first 160,000 s (Fig.1.7 and Eqs.(1.6)).

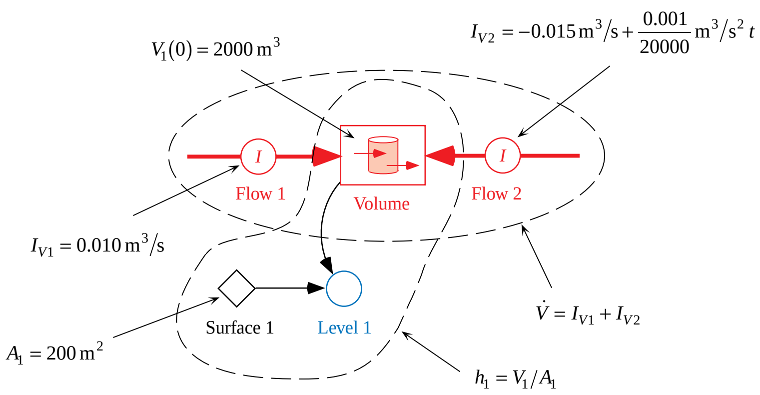


FIGURE 1.7. System dynamics diagram of the model of a lake fed by a stream and draining at the same time. Here, the law of balance of amount of water contains two flows instead of one. Moreover, the model contains a second relation which allows us to calculate the level of the water from the volume of water and the surface area of the lake.

There is a new relation showing up in the model diagrammed in Fig.1.7, namely the law relating the level of water to the volume and the surface area of the lake. The particular relation shown assumes straight vertical walls for the lake. After assembling all the relations and specifications of parameters, the complete set of equations making up our model of the lake looks as follows:

$$\begin{aligned}
&\dot{V}_1 = I_{V1} + I_{V2}\\
&V_1(0) = 2000\,\mathrm{m}^3\\
&I_{V1} = 0.010\,\mathrm{m}^3/\mathrm{s}\\
&I_{V2} = -0.015\,\mathrm{m}^3/\mathrm{s} + \frac{0.001}{20000}\,\mathrm{m}^3/\mathrm{s}^2\; t\\
&h_1 = V_1/A_1\\
&A_1 = 200\,\mathrm{m}^2
\end{aligned} \tag{1.6}$$

The solution of the set can be obtained with the same basic procedure as the one used so far. Instead of Eq.(1.4), we now have

$$V_{new} = V_{old} + \left(I_{V1} + I_{V2}\right)\Delta t \tag{1.7}$$

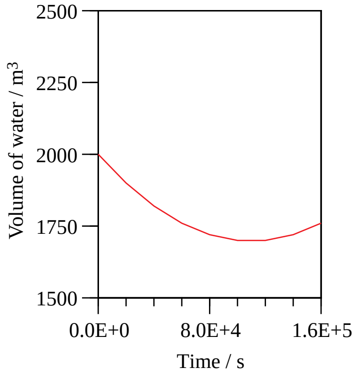


FIGURE 1.8. Numerical solution of Eqs.(1.6). Note that the volume of water in the lake decreases as long as the sum of the flows of water is negative.

Again we can use a spread sheet to do the actual computation; only, this time, the spread sheet is larger. The time step chosen is 20,000 s as in the second solution of Eqs.(1.5). The solution is represented graphically in Fig.1.8.

A More Realistic Representation of the Draining of the Lake

The special model produced so far certainly does not represent reality very well: the flows—especially the one draining the lake—can be specified at random to fit any desired behavior, which is hardly what we expect of a model. In nature, processes are determined by other quantities of the system under consideration or by other systems influencing it. In other words, flows are determined by the model itself.

In our case we expect the flow through the underground channel draining the first lake into the second one to depend on the circumstances of the system. The flow must in some way depend on the difference of the levels of water in the two lakes, and on the nature of the passage between the two and on the properties of water (Fig.1.9). In particular, if the water is at the same level in both lakes we should not see any flow at all. Since of the factors listed only the levels of the water can change, we will introduce them in a further refinement of the model. The properties of the channel and of the water will be expressed in terms of a constant factor (Flow constant). The simplest possible relation for Flow 2 is one for which I_{V2} is proportional to the difference of levels $h_1 - h_2$, with the flow constant k as the multiplying factor:

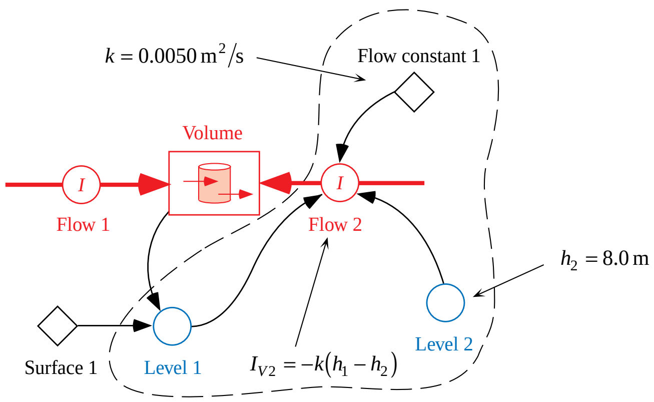


FIGURE 1.9. The extension of the previous model includes an expression for Flow 2 which depends upon the difference of water levels in the two lakes.

$$I_{V2} = -k(h_1 - h_2) \tag{1.8}$$

Before we go on to determine the behavior of the new model, we can use a general argument to determine the magnitude of the Flow constant k: as mentioned in the introduction to this example, it has been observed that the levels of the lakes stay constant if they measure 10 m and 8 m respectively, and if the stream feeding the first lake delivers 10 liters of water per second. Now, the level of the first lake can only stay constant if the inflow and the outflow of water exactly cancel. This requirement puts the value of I_{V2} at – 10 liters/s. For a difference of water levels of 2.0 m, k must therefore be 0.0050 m^2/s:

$$\begin{aligned} h_1 = const \quad &\Rightarrow \quad \dot{V}_1 = 0 \quad \Rightarrow \quad I_{V1} + I_{V2} = 0 \\ &\Rightarrow \quad 0.010\ \mathrm{m^3/s} + (-k \cdot 2.0\ \mathrm{m}) = 0 \\ &\Rightarrow \quad k = 0.0050\ \mathrm{m^2/s} \end{aligned}$$

If we replace Eq.(1.6)$_d$ by Eq.(1.8) and add values for h_2 and k, we arrive at the complete list of equations for the extended model (Eqs.(1.9). Naturally, giving h_2 a constant value reflects a fundamental flaw in the current model which will only be removed by considering the dynamics of the second lake as well. This will be done in the next subsection. Again, the equations can be solved by the spread sheet technique developed before. This time, however, the calculations are slightly more involved. The reason is to be found in the fact that the flows are not directly specified functions of time but rather depend upon quantities of the system which themselves are the result of the solution of the model equations. However, by progressing step by step in a spread sheet as in Table 1.1, the missing values are computed as they are needed.

$$\begin{aligned}
&\dot{V}_1 = I_{V1} + I_{V2}\\
&V_1(0) = 2000\,\mathrm{m}^3\\
&I_{V1} = 0.010\,\mathrm{m}^3/\mathrm{s}\\
&I_{V2} = -k(h_1 - h_2)\\
&h_1 = V_1/A_1\\
&A_1 = 200\,\mathrm{m}^2\\
&k = 0.0050\,\mathrm{m}^2/\mathrm{s}\\
&h_2 = 8.0\,\mathrm{m}
\end{aligned} \tag{1.9}$$

If we use the values for the levels and the feeding stream as detailed above, we should get a steady state, i.e., the levels of the lakes should not change. This can be verified by doing the appropriate calculations. However, it is certainly more interesting to do a calculation for which values are changing. Assume, therefore, that after a period of steady conditions, the level of the water in lake two suddenly drops to 5.0 m. As a consequence, the flow from the first to the second lake will increase, leading to a decrease of h_1. This in turn must decrease the magnitude of I_{V2}, which will make h_1 decrease more slowly, and so on. All in all we should expect h_1 to decrease to a level for which steady conditions are attained once more.

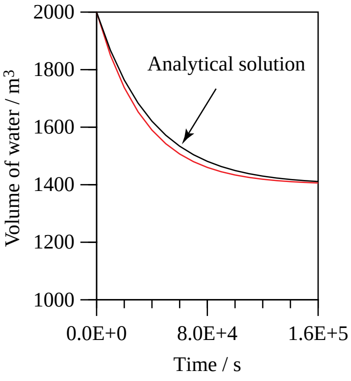


FIGURE 1.10. Numerical solution of Eqs.(1.9) compared to analytical solution. The level of water in the first lake reaches a new steady value of 7.0 m, again 2.0 m above the one of the second lake.

A Complete Model of Two Communicating Lakes

The model of the system of two lakes certainly cannot be complete without consideration of the dynamics of Lake 2 (Fig.1.1). This is accomplished simply by adding the law of balance of water for this partial system. Lake 2 is fed by the current of water flowing from the first lake, and it drains through the second underground channel into the open. Therefore, in a graphical representation of the model, we have to add another stock with two flows to the diagram of Fig.1.9. To complete the model we have to determine the new flows (Fig.1.11).

Since Flow 3 represents the water flowing from the first to the second lake, I_{V3} must simply be the negative of I_{V2}. For the flow draining Lake 2 (i.e., I_{V4}) we will use a relation very similar to the one written for I_{V2}; Eq.(1.8). Here, the difference of water levels is equal to the level of water in Lake 2:

$$I_{V4} = -k_2 h_2 \tag{1.10}$$

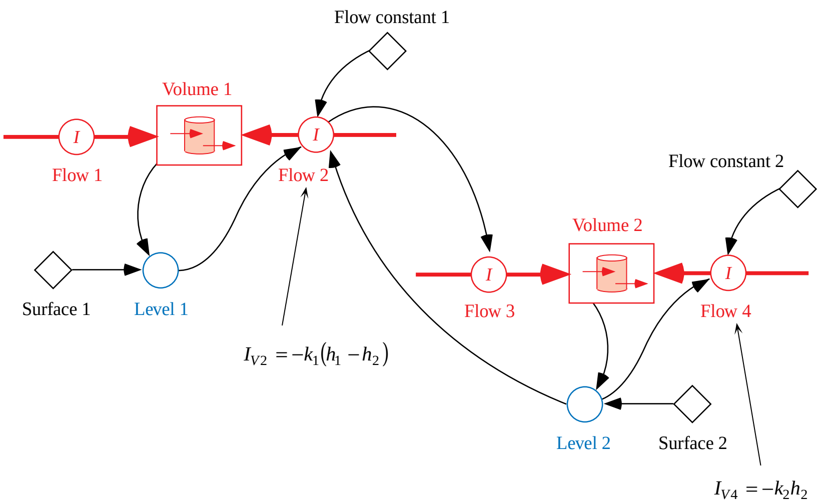


FIGURE 1.11. System dynamics model of two communicating lakes. Note the two stocks and associated flows which represent two laws of balance. Model equations:

$$\begin{aligned}
&\dot{V}_1 = I_{V1} + I_{V2}\\
&V_1(0) = 2000\,\mathrm{m}^3\\
&I_{V1} = 0.010\,\mathrm{m}^3/\mathrm{s}\\
&I_{V2} = -k_1(h_1 - h_2)\\
&\dot{V}_2 = I_{V3} + I_{V4}\\
&V_2(0) = 3200\,\mathrm{m}^3\\
&I_{V3} = -I_{V2}\\
&I_{V4} = -k_2 h_2\\
&h_1 = V_1/A_1\\
&A_1 = 200\,\mathrm{m}^2\\
&h_2 = V_2/A_2\\
&A_2 = 400\,\mathrm{m}^2\\
&k_1 = 0.0050\,\mathrm{m}^2/\mathrm{s}\\
&k_2 = 0.00125\,\mathrm{m}^2/\mathrm{s}
\end{aligned} \tag{1.11}$$

The second Flow constant k_2 may be determined in the same way as k_1. It is known that steady conditions prevail if the flow feeding the lakes is 10 liters/s, and the levels are 10 m and 8 m, respectively. For the steady state, the flows with respect to a reservoir have to balance. Therefore, we have $I_{V1} = I_{V2} = I_{V3} = I_{V4}$ (for the magnitudes of the fluxes). From this we conclude that

$$\begin{aligned} h_2 = const \quad &\Rightarrow \quad \dot{V}_2 = 0 \quad \Rightarrow \quad I_{V3} + I_{V4} = 0 \\ &\Rightarrow \quad 0.010\,\mathrm{m}^3/\mathrm{s} + \left(-k_2 \cdot 8.0\,\mathrm{m}\right) = 0 \\ &\Rightarrow \quad k_2 = 0.00125\,\mathrm{m}^2/\mathrm{s} \end{aligned}$$

The model is completed, and we can solve the equations. Basically, this works as before, i.e., we can use the spread sheet technique. Except for the number of operations to be performed, the computational technique is the same. It simply has to be applied to two laws of balance:

$$\begin{aligned} V_{1,new} &= V_{1,old} + \left(I_{V1} + I_{V2}\right)\Delta t \\ V_{2,new} &= V_{2,old} + \left(I_{V3} + I_{V4}\right)\Delta t \end{aligned} \tag{1.12}$$

Computations can be performed with all sorts of initial conditions or inflows. In Fig.1.12, the solution of the problem stated at the beginning of the section is shown in form of curves. The flow of water feeding the lakes is as specified in Fig.1.2, and the response of the lakes can be read from the curves representing the water levels.

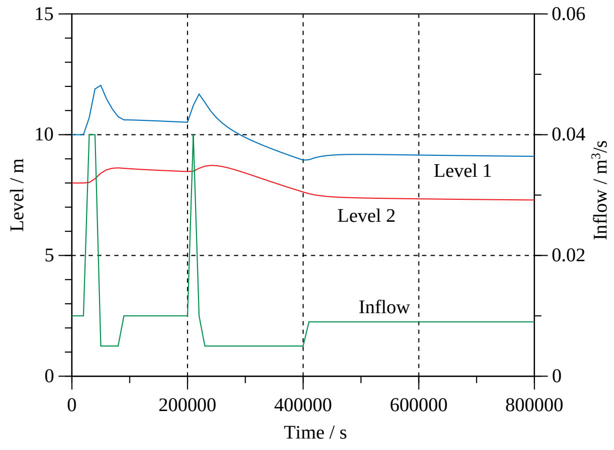


FIGURE 1.12. Behavior of the lakes according to the model represented by Fig.1.11 and Eqs.(1.11). The current of water feeding the lakes is as specified in Fig.1.2. Note that the lakes respond to an increase of the flow by increasing the level, but the response is delayed; the delay is larger for Lake 2. For a steady inflow which is smaller than the original one, the lakes find new steady values for the levels. The modeling and the simulation were done with the help of Stella II.

What Does the Model Teach Us About Hydraulic Processes?

Take a closer look at what we have assembled on the previous pages. The exercise should tell us much about the problem of hydraulic processes. Essentially, there are four types of relations to be found in the model that was constructed.

We started the modeling process by expressing our assumption that flows of water are responsible for changing the amount of water in each of the lakes. The formal rules behind these relations are called laws of balance. They are the general laws which relate processes (flows) to what happens to a system (changes of content). We will study laws of balance a little more closely in Section 1.2.

To flow through the underground channels, water needs a driving force. We know that pressure differences are the cause of the flow. Therefore we have to investigate pressure and pressure differences, and their role in hydraulic systems (Section 1.3).

Pressure differences and flows are related. In the case of fluids, the relationship may depend upon the channel and the properties of the fluids. We therefore have to learn more about how flows depend upon driving forces (Section 1.4).

Since we believe that the pressure differences are responsible for forcing the water through the underground channel between them, and since some of the pressure differences in the system depend upon water levels in the lakes, we should be able to calculate these levels. Naturally, they are related to the amount of water in a lake and the shape of the body of the lake (Section 1.5).

Q

1. If you had to take into account evaporation of water from the surface of Lake 1—at a constant rate proportional to the surface area—how would you have to change the model diagrams in Figs.1.7 and 1.9?
2. Does the addition of evaporation change the nature of the relationships between volume, surface, and level (Fig.1.7) or between flow and differences of levels (Fig.1.9) in any way? Does it change the behavior of the system?
3. Consider two communicating lakes without inflow and outflow. When does the current of water between them stop? What condition is attained when nothing changes anymore?
4. Can you think of a numerical scheme which yields accurate results for the problem of Eqs.(1.5)? Would this scheme yield accurate results for more general flows?
5. What is the graphical representation of the relation between flow and difference of levels used for the models shown in Figs.1.9 and 1.11? Do you expect the relation between flow through a channel or pipe always to be of this form? Do you know the equivalent diagram used in electricity?

A

1. You first have to add another flow representing evaporation to the structure of stock and flows in the diagram(s). The you have to express the flow as the product of some constant and the quantity Surface 1.
2. The particular laws from which we calculate the levels and the flows have nothing to do with how many flows there are in the structure of stocks and flows. However, changing the flows naturally changes the behavior of the system.
3. Connecting different types of containers always leads to the same result: in the end, the levels of fluid must have become equal. This is reflected in our model: we can see in Eqs.(1.11) that the flow I_{V2} (which is the same as I_{V3}) stops when this condition is met.
4. For Eqs.(1.5) we could choose a method which uses the values of the flows at the beginning and the end of the time step Δt and calculates the average value. Multiplying this average flow for Δt by the time step yields a correct value for the volume of water exchanged as long as the flows are linear functions of time. For flows which are more general functions of time this scheme is again inaccurate.
5. A graph of I_V versus the difference of levels yields a straight line going through the origin of the graph; you can see this by inspecting the equation for I_{V2} (Eq.$(1.11)_d$). We only get this linear form if the flow is independent of the channel or pipe, and the fluid properties. In electricity, the equivalent graphs are called *I-U* (current-voltage) curves or lines. (See Chapter 2).

1.2 Balancing Amounts of Fluids

Storage and flow of fluids are at the heart of hydraulic phenomena. Since the flow of a fluid is responsible for the change of the amount of fluid contained in a system, there must be a relation between flows and these changes; the relation is called a *law of balance*. The art of balancing amounts of water requires us to introduce abstract quantities with which we measure flows and amounts. Then, related quantities are defined and with their help the law of balance is expressed. Here we will get to know a graphical method for dealing with the relations between flows and change.

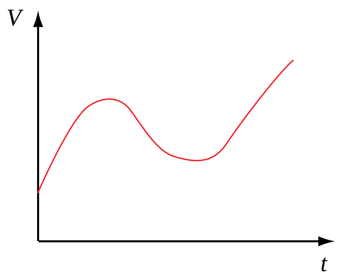


FIGURE 1.13. The volume of a fluid stored in a system changes in time as the result of a dynamical process. The graph of $V(t)$ lets us visualize most easily what is happening to the quantity V.

System Content: Volume and Rate of Change of Volume

We need an abstract quantity to measure the amount of a fluid contained in a system. Nature offers three measures: *amount of substance* which used by chemists, *mass* which is mostly used by mechanical engineers, and *volume*. Under the simplest of circumstances these measures are related by constant factors making them practically equivalent. If there are no chemical reactions, and if the fluid is incompressible, i.e., if the volume of a given amount of fluid cannot be changed, the factors relating amount of substance and mass, and mass and volume, are constant. For reasons of simplicity we shall mostly choose the volume of the fluid as the measure of its amount. Since it is of interest as well, we shall sometimes use the mass of a fluid when convenient. The relation between mass and volume is

$$m = \rho V \tag{1.13}$$

where ρ is the *density* of the fluid. We measure volume in m^3, mass in kg, and density in kg/m^3.

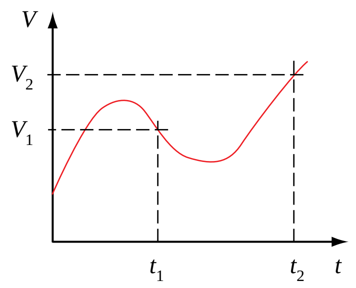


FIGURE 1.14. We can read values of V for specific moments directly from the graph. Also, the change of V, abbreviated by ΔV, can be read off the vertical axis.

Since we are dealing with dynamical processes, the volume of fluid in a system must change with time. Therefore, we do not need only single values of this quantity. Rather, we are interested in the *volume as a function of time*. Such functions can be represented in various ways such as equations, tables, and graphs, where graphs convey the clearest image of the quantity as it changes in the course of time (Fig.1.13).

Change and rate of change. The graph of $V(t)$ also contains all the other information important to a description of what is going on with the amount of fluid stored in the system. First, we can read from it values of V for specific moments in time. Second, we can determine the change of volume for a given period of time lasting from t_1 to t_2 (Fig.1.14). The *difference of volume* for two points in time is defined as

$$\Delta V = V_2 - V_1 \tag{1.14}$$

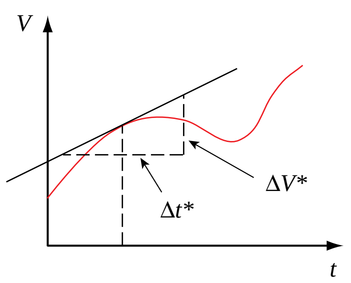


FIGURE 1.15. The quantity measuring how fast the volume changes at an instant—called the *rate of change of volume*—is visualized by how steep the curve is. This in turn is determined by the slope of the tangent to the curve for a given instant.

It is always the later value minus the earlier one; also, it is independent of what happens to the volume in the time span between the two instants t_1 and t_2. Third, and most important, the graph also tells us *how fast the volume is changing* at any given moment. Obviously, the graph of $V(t)$ also contains this information. If we look at the curve representing the volume as a function of time we can see how fast V is changing from how steep the curve is at that point in time. The steeper it is, the faster the change. Now we use the graphical representation of V to define what we mean by *how fast* the quantity changes. We can measure how steep the curve is at a point by drawing the tangent to the curve (Fig.1.15) and determining the slope of this straight line. The *slope of the tangent* is said to measure *the time rate of change* of the quantity V at the given point. The slope of the tangent can be determined from any rectangular triangle with horizontal and vertical legs having the tangent as the hypotenuse. It is simply $\Delta V^*/\Delta t^*$ as seen in Fig.1.15, and has units m^3/s. As mentioned before, the rate of change of V is denoted by the symbol $\dot{V}$. There are other methods of determining the rate of change of V from the knowledge of $V(t)$—such as numerical and formal ones—but this one is the most easily visualized.

A simple numerical method to determine the rate of change of volume is the following. Assume that we know the values of V at as many points t as we wish. We may know a formal representation of the function—an equation for $V(t)$—or we may have a table of values with many entries for finely spaced points in time. The method also can be explained in graphical terms (Fig.1.16). Imagine two points separated by a small interval Δt. The associated values of V are supposed to be known. We can approximate the slope of the curve in the narrow interval by the line connecting the two points on the curve (the secant). The slope of the secant is determined easily:

$$\dot{V} \approx \frac{\Delta V}{\Delta t} = \frac{V_2 - V_1}{t_2 - t_1} \qquad \textbf{(1.15)}$$

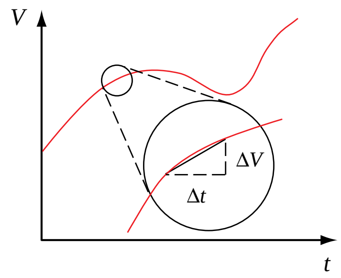


FIGURE 1.16. The slope of a curve can be approximated by the slope of a straight line connecting two neighboring points. The smaller the interval Δt, the better the approximation.

The quantity $\Delta V/\Delta t$ is called the *average rate of change* for the interval Δt. In a table of values the average rate of change is determined easily for each interval according to the rule of Eq.(1.15). Because of the similarity with calculating the ratio of the difference of volume and a time span Δt, we also use the symbol dV/dt for the rate of change of volume:

$$\dot{V} \equiv dV/dt \qquad \textbf{(1.16)}$$

Naturally, the rate of change of V can be determined for any point in time for a smooth curve. If we do this graphically as described before for a number of points, we can transfer the values of dV/dt to a table from which we can create a graph of the rate of change (Fig.1.17).

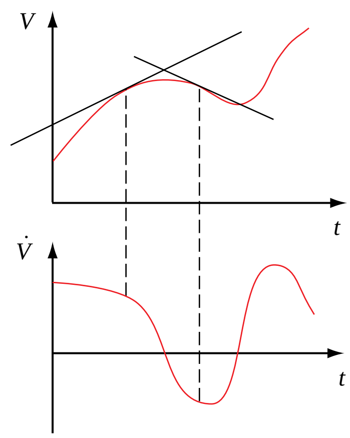


FIGURE 1.17. The rate of change of volume is itself a function of time. It can be found graphically from the curve $V(t)$ by drawing several tangents and determining their slopes. Note that the rate of change is negative for sections of $V(t)$ where V is decreasing, i.e., where the tangent is sloping.

Calculating the volume from its rate of change. We have just learned how to calculate changes of volume and the rate of change dV/dt from the information contained in $V(t)$. We can also perform the reverse process: if we know the rate of change, we can calculate changes of volume. A diagram shows how this works. First, we consider the case of constant rate of change. In this case the change of volume must be equal to the product of rate of change and time span Δt:

$$\Delta V = \dot{V}\Delta t \qquad \textbf{(1.17)}$$

Graphically speaking, ΔV is equal to the area of the rectangle between the straight line $V(t)$ and the t–axis with width Δt. (See the left diagram of Fig.1.18.) In general, for arbitrary functions $V(t)$, we have to determine the area between the curve $V(t)$ and the t–axis in the chosen time interval. (See the diagram on the right of Fig.1.18.) One approximate method of determining the area bounded by a curve is discussed below.

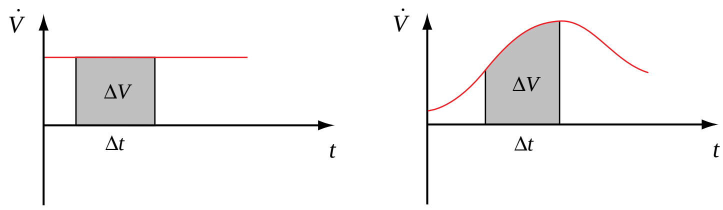


FIGURE 1.18. The change of volume can be determined from the diagram of dV/dt as a function of time. In general, the change of V in time span Δt is equal to the are between the curve of $V(t)$ and the t–axis for the time interval.

Assume that a process specifies the rate of change of the volume of a fluid as a function of time. From this we can recreate the volume as a function of time if we also know the value of V at the beginning, i.e., the initial value V_o. The procedure works as follows (Fig.1.19). In the diagram for dV/dt, choose a (small) time step Δt and determine the

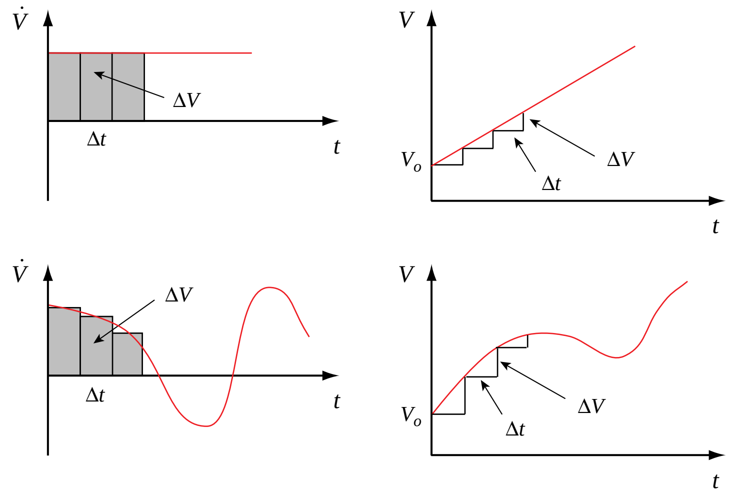


FIGURE 1.19. An area between the curve representing the rate of change of volume and the t–axis in an interval Δt yields the change of V for the particular interval. By dividing the time axis into many small intervals, and calculating the areas of narrow rectangles, we can construct $V(t)$ if we know the initial value V_o.

area of a rectangle with a height equal to the average value of the rate of change for the first interval. This approximately represents the change of V during Δt:

$$\Delta V \approx \dot{V}\Delta t \tag{1.18}$$

(Note that this equation returns the correct value of the change of V only if the volume is a linear function of time.) Now move to the diagram which will contain $V(t)$ and indicate V_o on the vertical axis. Then move horizontally and vertically by a distance Δt and ΔV, respectively. Repeat the process for consecutive rectangles in the diagram for dV/dt, and you will trace an approximation to the function $V(t)$.

Q

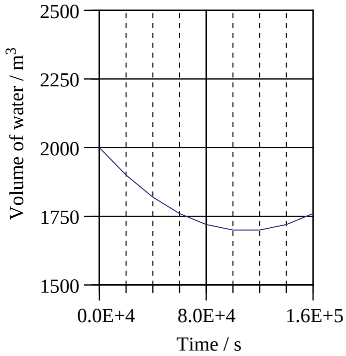
1. What is the meaning of volume, change of volume, and rate of change of volume? How can you determine the rate of change from $V(t)$?
2. How do you calculate the change of volume for a time span if the rate of change is known? What additional value needs to be known if you want to determine the volume as a function of time?
3. If the rate of change of volume is negative, what sign should the area between the curve of dV/dt and the t-axis have?

A

1. Volume measures the quantity of a fluid inside a system; the change of volume is the difference of a value of volume at t_2 and one at an earlier time t_1; the rate of change measures how fast the volume is changing in a process. The latter is equal to the slope of the tangent to $V(t)$.
2. The change of volume is equal to the area between the curve dV/dt and the t–axis for the interval Δt. In addition to the rate of change as a function of time we need to know the initial value of the volume.
3. For negative rates of change, the curve is below the axis, and the change is negative.

EXAMPLE 1.1. Graphical determination of rate of change of volume.

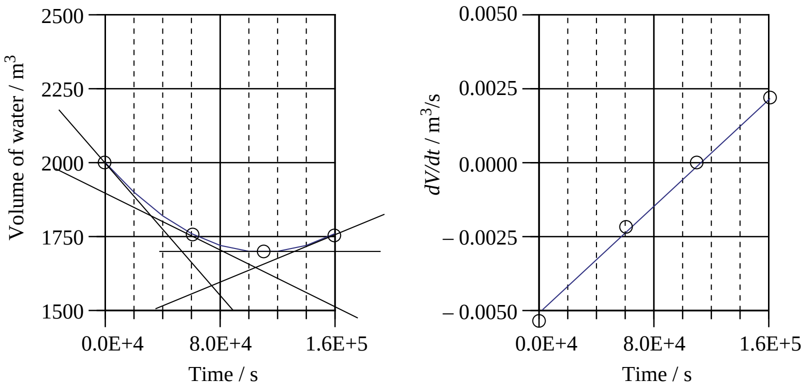
The volume of water in a lake is known in graphical form as a function of time. a) Determine the mass of the fluid at t = $6.0 \cdot 10^4$ s. b) Determine the rate of change of the volume for a few points by graphical means and transfer them to a graph of *dV/dt* versus *t*. Is there a simple function which fits the points? c) Explain the meaning of the different signs of *dV/dt*, and of the minimum of the curve *V*(*t*).



SOLUTION: a) The mass for a given volume is calculated with the help of Eq.(1.13). The density of water is 1000 kg/m^3. With a volume of 1750 m^3 read off the graph we get

$$m\left(t = 6.0 \cdot 10^4\,\mathrm{s}\right) = \rho V\left(t = 6.0 \cdot 10^4\,\mathrm{s}\right) = 1000\,\mathrm{kg/m^3} \cdot 1750\,\mathrm{m^3} = 1.75 \cdot 10^6\,\mathrm{kg}$$

b) Draw tangents to the curve at a few points as carefully as possible. For the one at $t = 0$ s, the slope is determined as follows: for a base line of $\Delta t = 9.0 \cdot 10^4$s, the change of V is $\Delta V = -500$ m^3. This yields a value of $-5.5 \cdot 10^{-3}$m^3/s. A straight line seems to fit the data points in the second graph very well.



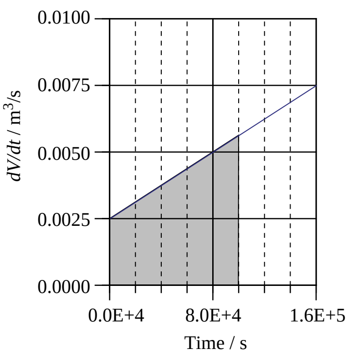
c) The volume decreases for the first $1.1 \cdot 10^5$s corresponding to negative values of the rate of change. After this, the volume increases which means that the rate of change must be positive. At the point in time where *V*(*t*) has a minimum, the rate of change is zero.

■

EXAMPLE 1.2. Finding the volume from the rate of change.

The rate of change of volume of water in a lake is as shown in the figure; it increases linearly from 0.0025 m^3/s to 0.0075 m^3/s in $1.6o \cdot 10^5$ s. Determine the volume analytically as a function of time. The initial value is $V_o = 1000$ m^3.

SOLUTION: We can easily solve the problem formally since the rate of change is a linear function of time. Therefore, we can find the corresponding formula for the area between *dV/dt* and the *t*–axis. This area represents the change of volume from 0 s to *t*.



The rate of change is the following function of time:

$$\dot{V}(t) = a + bt \quad , \quad a = 0.0025\,\mathrm{m^3/s} \quad , \quad b = 3.125 \cdot 10^{-8}\,\mathrm{m^3/s^2}$$

The area to be calculated is that of a trapeze going from 0 s to *t*. The volume at t is found by adding the initial volume V_o:

$$\Delta V(t) = \frac{1}{2}\left(\dot{V}(0) + \dot{V}(t)\right)\Delta t = \frac{1}{2}(a + a + bt)t = at + \frac{1}{2}bt^2 \quad \Rightarrow \quad V(t) = V_o + at + \frac{1}{2}bt^2$$

■

Flows: Currents and Amounts Exchanged

Volume and rate of change of volume are the quantities that have to do with the system content only. For a complete description of dynamical processes we also have to be able to describe the interaction of the system with the environment which is done with the help of the flows of fluids.

Flows, currents, fluxes. Flows of fluids are easily visualized. When we stand next to streams and rivers, or pipes leading into fountains, we can get a good feeling for the magnitude of a current (Fig.1.20). The abstract quantity measuring a current is called a *flux*. A *flux of volume* is given the symbol I_V, and its units are m^3/s. Fluxes are measured at surfaces cutting through currents. The surface is given an orientation indicated by a vector perpendicular to the surface area, and a flow going in the direction of this vector is given a positive flux (Fig.1.21); a flow going in the opposite direction is measured in terms of a negative flux. Note that the closed surface of a body or a region of space can be given an orientation either outward or inward. There are different customs in this respect. Physicists and mathematicians often take the outward normal as the positive direction, in engineering it often is the inward direction which is given a positive value. We shall choose the latter convention, which means that the flux associated with a flow into a system is counted as a positive quantity.

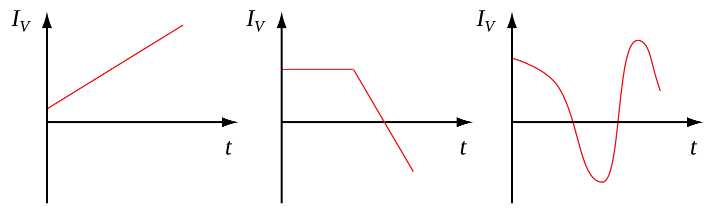


FIGURE 1.20. Examples of fluxes as functions of time. A flux measures the magnitude of a flow at a given surface where the sign of the flux denotes the direction of flow with respect to the orientation of the surface.

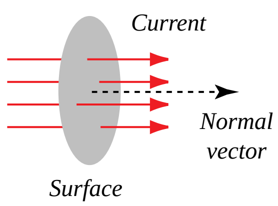


FIGURE 1.21. Examples of fluxes as functions of time. A flux measures the magnitude of a flow at a given surface where the sign of the flux denotes the direction of flow with respect to the orientation of the surface.

The quantity exchanged with a flow. Knowing the flux of a current of fluid allows us to calculate how much is flowing across a surface in a given time span. If the flux is constant in time, this is particularly simple. The quantity V_e exchange with a current of volume flux I_V is given by

$$V_e\left(t_1 \to t_2\right) = I_V \Delta t \tag{1.19}$$

If the flux is variable, the determination of the amount exchanged is performed in exactly the same manner how we determine changes of volume from the rate of change (Fig.1.18). We simply have to draw the flux as a function of time and calculate the area between the curve $I_V(t)$ and the t-axis for the period of time considered in the problem (Fig.1.22).

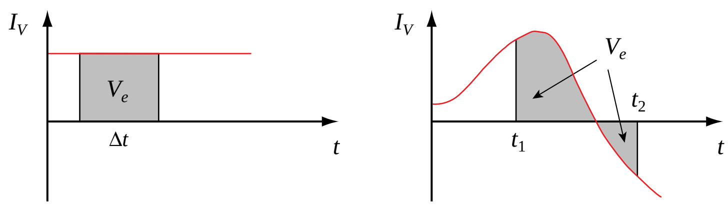


FIGURE 1.22. The same mathematical operation which is used to calculate changes of volume also yields the volume exchanged with a current. Instead of the rate of change we have the flux as a function of time.

Measurement of fluxes. There are several ways of measuring fluxes of fluids in practise. The simplest, but not necessarily the most enlightening, is this. We can let a current flow for short amounts of time, every time measuring how much water has been transported. The amount transported divided by the time span used yields an approximate value, i.e., the average value, of the flux. There are more direct ways, however, to measure a flux in the laboratory. Certain sensors directly detect the flow of a fluid, rather than amounts transported. One such example is a sensor using the principle of electromagnetic induction (Chapter 33).

Accounting: The Law of Balance of Volume

Storage and flow are the two phenomena which combine to make dynamic processes possible. We know that there exists a relation between flows of a fluid with respect to a fluid store and how the volume of the fluid is changing in that storage device. This relation is called the *law of balance of volume* (of a fluid).

Experience tells us that all the flows together are responsible for the change of the amount of fluid in a store. The larger the sum of all currents, the faster the volume of fluid must change. This observation may be summarized as follows:

■ The sum of all fluxes with respect to the store tells us how fast the stored amount is changing. In formal terms, this means that the rate of change of volume of a system must be equal to the sum of all fluxes of volume with respect to the system:

$$\frac{dV}{dt} = \sum_{i=1}^{N} I_{Vi} \tag{1.20}$$

Alternatively, the law of balance of volume also can be expressed in terms of the change of volume and the net volume exchanged as a result of the fluxes:

$$\Delta V = \sum_{i=1}^{N} V_{ei} \tag{1.21}$$

This form holds for a certain time span Δt, whereas Eq.(1.20) holds for any moment in time.

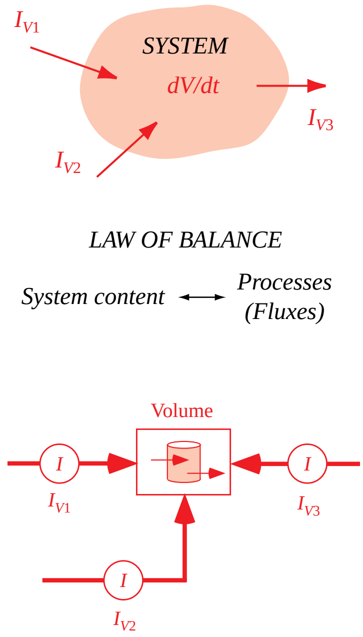


FIGURE 1.23. Laws of balance combine the two sides of dynamical processes: storage and transport processes. They tell us how the processes—expressed in terms of fluxes—determine what is happening to the system content—expressed in terms of the rate of change of the content. The lower part of the figure is the system dynamics representation of the law of balance.

There are several important points to be noted about a law of balance such as the one in Eq.(1.20):

- ► The fluxes of currents flowing into and out of the system (and the associated amounts which are exchanged) have to be given the proper signs: positive for inflow, negative for outflow.
- ► A law of balance is not a definition of a current: *dV/dt* is *not* a current I_V. In fact, *dV/dt* is not even the sum of all fluxes (it is not identical to the sum of the fluxes), its value is only *equal to* this quantity.
- ► A law of balance can be used in two ways. First, if the sum of all fluxes is known we also know the rate of change of the system content. Second, if the rate of change of volume of a system is known, and if we know all but one current, the missing flux can be calculated.
- ► By itself, a law of balance is not of much use. Only if it is combined with special knowledge about the fluxes can it be used for calculations.
- ► The law containing the rate of change and the fluxes is called the *instantaneous* form of the law of balance; it holds for every moment in time. Its counterpart, i.e., Eq.(1.21), is called the *integrated* form of the law.

1. Why isn't a current a rate of change? Why isn't the volume exchanged a change of volume?
2. Describe the current whose flux is shown in the second diagram of Fig. 1.20. What does a negative flux mean?
3. The rate of change of volume is known. What does this tell you about the fluxes?

Q

1. A current is a fundamental quantity introduced for describing a transport process. A rate of change is a quantity which describes a system and its content. The same can be said about the integrated quantities.
2. First the magnitude of the current is constant, and the flow is in the direction indicating the orientation of the surface are through which the current flows. Then the magnitude decreases linearly. After it has become zero, the flux becomes zero, i.e., the flow reverses its direction.
3. If the rate of change is known, only the sum of the fluxes can be determined from the law of balance. Information on single flows must be obtained by special laws for the flows.

A

EXAMPLE 1.3. A simple case of balancing the volume of water.

Consider a water container with an inlet and an outlet. The flow at the inlet is constant and equal to 10 liters/s. The volume is known to change at a rate of – 13 liters/s. a) Determine the flux of volume associate with the outlet; express it in terms of the flux of mass. b) How much water is exchanged with the current flowing out of the container in the first minute? c) Determine the volume as a function of time. The initial volume is 20 m^3.

SOLUTION: a) The law of balance of volume in its instantaneous form lets us determine the missing flux:

$$\dot{V} = I_{V1} + I_{V2} \quad \Rightarrow \quad I_{V2} = \dot{V} - I_{V1} = -13 \cdot 10^{-3}\,\mathrm{m}^3/\mathrm{s} - 10 \cdot 10^{-3}\,\mathrm{m}^3/\mathrm{s} = -23 \cdot 10^{-3}\,\mathrm{m}^3/\mathrm{s}$$

$$I_m = \rho I_V \quad \Rightarrow \quad I_{m2} = 1000\,\mathrm{kg}/\mathrm{m}^3 \cdot \left(-23 \cdot 10^{-3}\,\mathrm{m}^3/\mathrm{s}\right) = 23\,\mathrm{kg/s}$$

b) The amount exchanged is calculated according to Eq.(1.19):

$$V_{e2} = I_{V2}\Delta t = -23 \cdot 10^{-3}\,\mathrm{m}^3/\mathrm{s} \cdot 60s = -1.38\,\mathrm{m}^3$$

c) The change of volume can be calculate in a couple of ways. The most direct is

$$V(t) = V_o + \Delta V_{0\to t} = V_o + \dot{V}(t-0) = 20\mathrm{m}^3 - 13 \cdot 10^{-3}\,\mathrm{m}^3/\mathrm{s} \cdot t$$

■

1.3 Pressure and the Hydraulic Driving Force

Normally, water or oil would not flow by themselves. This simple observation means that other processes are responsible for driving the one we are interested in. We say that they set up a driving force for the fluids to flow.

Examples of Driving Forces

To be precise, there are two things fluids do not do by themselves here at the surface of the Earth: they do not flow uphill, and they do not flow horizontally through pipes and channels, if unaided. In the case of horizontal flow we say that fluid friction is the reason why a flow must be forced (Section 1.4).

There are several concrete circumstances or setups which are used to force fluid flow where this is necessary. In nature, we observe mostly two cases. First, water flows downhill—aided by gravity—leading to other flows. Second, in the atmosphere, areas of high pressure build up and the air flows to locations where the pressure is lower. In technical settings we can use containers where the pressure is higher at the bottom of the fluid, pressure vessels, and pumps (Fig.1.24). These observations suggest that pressure differences are the cause of fluid flow which would not take place otherwise. Again, the example of two communicating fluid tanks is most helpful in convincing us of this. (See the first two entries in Table I.1 in the Introduction.) The fluid will only flow through the connecting pipe if there is a difference of levels (pressures) from one end of the pipe to the other. The process of flow stops when the pressure difference has become zero.

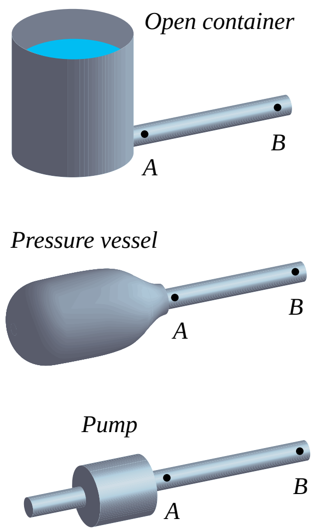


FIGURE 1.24. For the fluid to flow through the pipe from A to B, the pressure has to be higher at A. A container with fluid, a pressure vessel (containing a fluid a high pressure), or a pump can be used to set up such a pressure difference.

The Pressure of Fluids

Pressure is one of these fundamental physical quantities for which we have good everyday knowledge. We know this quantity from the atmosphere and the lakes. In the atmosphere, pressure changes from point to point, and in lakes the pressure of the water increases downward. Pressure is a quantity describing a hydraulic state of a fluid at every point inside.

Pressure is measured in Pascal (Pa). Going vertically down in the water of a lake here on our planet makes the pressure increase by about 10^4 Pa (Fig.1.25). The pressure of the air in the atmosphere is around 10^5 Pa at sea level. The pressure increases linearly downward in an incompressible fluid. A proof will be given in Chapter 29:

$$P = P_a + \rho g z \tag{1.22}$$

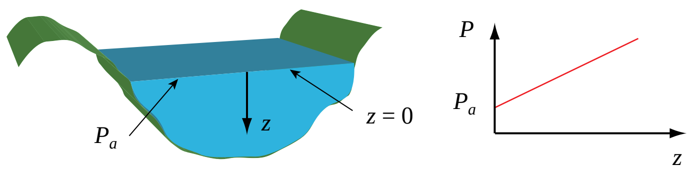


FIGURE 1.25. In incompressible fluids at the surface of the Earth, the pressure increases linearly downward. P_a is the ambient pressure (air pressure) at the surface of the lake.

Here, g measures the strength of the gravitational field. Fluid levels in vertical and U-tubes make for one of the simplest devices for measuring pressures (Fig.1.26).

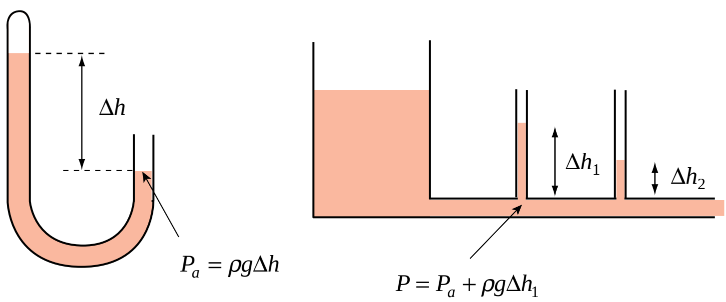


FIGURE 1.26. Tubes filled with liquids such as water or mercury serve as measuring devices for pressure. There is no air above the fluid in the closed section of the U-tube.

Pressure Differences in Hydraulic Circuits

If we want to be able to specify the conditions for the flow of fluids we must be able to determine pressures and pressure differences along the paths taken by the fluids. Take as an example the hydraulic setup shown in (Fig.1.27).

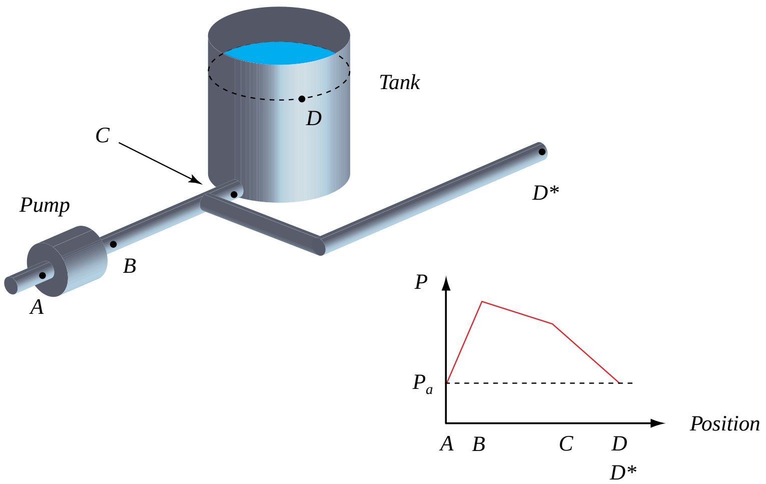


FIGURE 1.27. Knowing pressures and pressure differences along flows is very important in modeling hydraulic systems. Here, a few points have been indicated along the fluid paths. The pipes are open at *A* and *D**, as is the tank. The tank is filled up to the level indicated by point *D*. The graph next to the picture displays a qualitative sketch of the pressure at different points in the fluid. The air pressure is taken to be P_a.

A fluid such as oil flows from *A* to *B* through a pump, from there through a first pipe and into the tank. At *C*, the pipe branches off, and there is a second pipe at ground level leaving from point *C*.

We want to determine the values of the pressure at the points indicated in the figure. First, note that the pressure must be equal at *A*, *D*, and *D**, since there the fluid communicated with the air and has the same pressure as the surrounding air (P_a). In fact, we can imagine a closed hydraulic loop from *A* to *B* to *C* up to *D* and back through the air to *A*. Naturally, we expect to be back at the same hydraulic level, i.e., the pressure, once we have completed the loop. Then there is a second branch from *C* to *D** which is in parallel to the one from *C* up to *D* in the tank. Therefore, there is a second closed loop from *A* through *B*, *C*, *D**, and back to *A*. We can even identify a third closed loop from *C* to *D*, and from there via the air to *D**. From what we have said we expect an important rule to hold for closed hydraulic circuits:

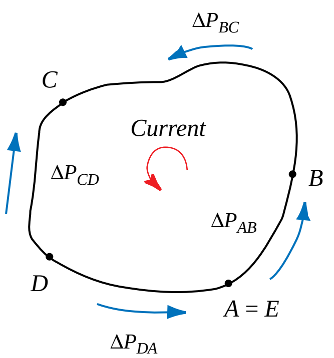


FIGURE 1.28. Pressure differences along a closed hydraulic loop add up to zero. A blue arrow pointing in the direction of flow denotes decreasing pressure, one going against the current denotes increasing pressure.

- No matter how simple or how complicated a closed loop, the sum of all pressure differences between points along the loop must be zero:

$$0 = \sum_{i=1}^{N} \Delta P_i \qquad (1.23)$$

We will find an equivalent rule which holds for electric circuits. There it is called Kirchhoff's second rule (Chapter 2).

Now let us have a closer look at the segments of the paths in our example. The fluid is taken up—maybe from a shallow container exposed to the air—at pressure P_a. If the segment of pipe leading to *A* is short, we expect the pressure at A to be P_a as well. The purpose of the pump is to increase the pressure of the fluid to a value at *B* such that it can flow as it is supposed to. After *B*, the pressure of the fluid is expected to drop because of hydraulic friction in the pipe leading to *C*. From *C* to *D*, we go vertically up in a fluid column. Since the pressure increases if we go down in a fluid, we know that here the pressure must drop once more, this time back to air pressure. The same drop of pressure must occur from C to D^* because at D^* we are back at P_a. We recognize that the pressure differences in parallel branches must be equal.

Q

1. A device such as the U-tube of Fig.1.26—filled with mercury—makes a simple tool for measuring air pressure. If the pressure of the air is 0.90 bar (1 bar equals 10^5 Pa), how high should the mercury column on the left rise compared to the one on the right?
2. How large is the pressure drop from C to D^* in Fig.1.27? Point D is at height h above ground, and the density of the fluid is ρ. Which laws or relations are necessary for determining the answer?

A

1. Since the pressure is equal at equal levels in the mercury, the pressure at depth Δh in the left column must be equal to the pressure of the air. With a density of 13,600 kg/m^3, we get $\Delta h = 0.67$ m. (With $g = 9.81$ N/kg.)
2. According to the loop rule in Eq.(1.23) the pressure difference from C to D^* must be equal to the one from C to D. Finally, with Eq.(1.22), we have $\Delta P_{C-D} = -\rho g|\Delta h|$. Note that the difference is negative.

1.4 Currents: Driving Forces and Resistance

The pressure can change for numerous reasons in a fluid, one of which is fluid friction. When fluids flow steadily and horizontally, it is found that the pressure decreases in the direction of flow (Fig.1.26 on the right). We say that a pressure gradient builds up as a result of hydraulic resistance when a flow is forced through a pipe (Fig.1.29).

Current–Pressure Characteristic Curves

We expect a relation between the volume flux through the pipe and the pressure drop across the length of the pipe. Naturally, this relation should be different for different circumstances. It depends upon the fluid property which leads to fluid friction, on the

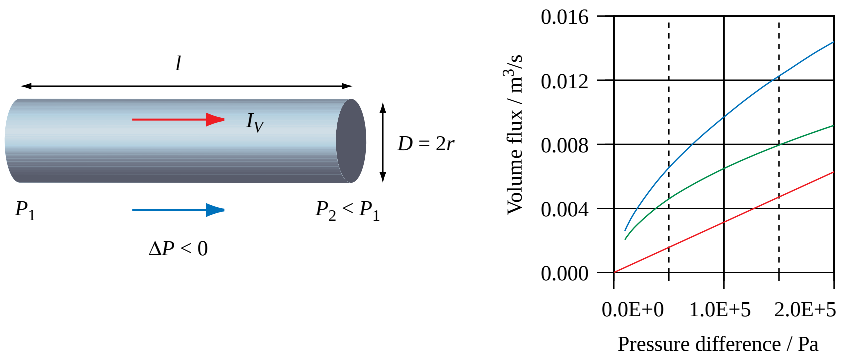


FIGURE 1.29. $I_V - \Delta P$ characteristic curves for different steady flows through the same pipe. (Length: 10.0 m, radius: 0.020 m.) The lower straight line is for a fairly viscous oil (viscosity: 0.20 Pa·s, density: 800 kg/m^3) for laminar flow. The upper two are for water (viscosity: 0.0010 Pa·s) for turbulent flow. The curve in the middle is for a rough pipe, the upper one for a smooth pipe.

size of the pipe (length and radius), and the type of flow (laminar or turbulent, or flow through a sand filled pipe). The relation can be displayed in diagram showing volume flux as a function of pressure drop. It is called an $I_V - \Delta P$ *characteristic curve*.

Driving Force and Hydraulic Resistance

Fig.1.29 shows three examples of $I_V - \Delta P$ characteristic curve for liquids flowing steadily through a pipe. There is a simple graphic way of describing the relationship between flux and pressure drop. Since the phenomenon is caused by friction which makes the pressure difference necessary, we say that the current is the result of the interplay between a *driving force* and a *resistance*. The larger the driving force, i.e., the pressure difference, the larger the flux, and the larger the resistance the smaller the current. Alternatively, instead of talking about a resistance, we can introduce a *conductance*. The current is imagined to be the result of a driving force and a factor telling us how easy it is to transfer the fluid. This image may be expressed as follows:

- If fluid friction leads to a pressure drop in the direction of flow, we may express the current of fluid in terms of the driving force, i.e., the pressure difference along the flow as a result of friction, and a factor called the *conductance* G_V:

$$I_V = -G_V \Delta P_R \qquad \textbf{(1.24)}$$

or, equivalently,

$$I_V = -\frac{1}{R_V} \Delta P_R \qquad \textbf{(1.25)}$$

where $R_V = 1/G_V$ is the called the *hydraulic resistance*.

ΔP_R is the pressure drop (in the direction of flow) as the result of the resistance. Obviously, the resistance is measured in $\mathrm{Pa \cdot s/m^3}$.

We can now answer the question if there are conditions in horizontal steady flow where there is no need for a pressure difference as a driving force. We can see from Eq.(1.25) that the hydraulic resistance would have to be zero. Fluids with this property do not exist, but we can at least imagine them. Such fluids would be called *ideal*.

Note that the flow which has a linear characteristic curve in Fig.1.29 (the lowermost straight line) has a constant conductance or resistance, while for the other two flows the resistance varies as a function of the difference of pressures.

Production of Heat in Flow With Friction

The pressure may change in a fluid for many different reasons. Therefore it is important to characterize the phenomenon described in this section more carefully. First we note that the pressure difference in the direction of flow is negative—the fluid flows “downhill”. This, however, does not suffice for a complete physical description. Fluids flowing from points of high to points of low pressure may be found in different situations such as flow through turbines (Section 1.6). Also, a pressure drop may result in the speeding up of a fluid (Chapter 3).

In resistive flow, on the other hand, the only consequence of the process is the production of heat. Therefore, we may say that fluid resistance is associated with *dissipation* (another term for production of heat; see Chapters 8–10). Consequently, ideal fluids do not produce any heat when flowing.

Laminar Flow and the Law of Hagen and Poiseuille

The type of flow has a strong influence upon the relation between flux and pressure drop. (See the graph in Fig.1.29.) We shall restrict our attention to flow through pipes with constant diameter, and to steady flow. Steady means that flow properties do not change with time. In an empty pipe, the flow of a fluid is laminar if the flow speed is low. By the way, the average flow speed $\bar{v}$ for a cross section is related to the flux by:

$$I_V = A\bar{v} \tag{1.26}$$

Laminar flow is most easily observed in the first few centimeters of smoke rising from a burning cigarette. If we could see stream lines we would see that they are all parallel; put differently, adjacent packets of the fluid do not mix. If the speed of flow is increased, there is a relatively sharp transition to turbulent, chaotic motion where the fluid is vigorously mixed. If everything else is kept constant, the onset of turbulence depends upon the viscosity of the fluid: the more viscous, the longer the flow stays laminar. Water has such a low viscosity that its flow through an otherwise empty pipe is practically turbulent all the time for reasonable values of the speed of flow. Some values of the viscosity of different fluids are listed in Table 1.2.

TABLE 1.2. **Viscosity at 20°C**

Fluid	Viscosity η / Pa·s
Castor oil	0.99
Glycerine	1.48
Olive oil	0.081
Mercury	0.00155
Water	0.00100

Experiments show that in laminar flow the relation between flux and pressure drop is linear as shown in the lower line in Fig.1.29. The conductance is expected to depend upon the radius and the length of the pipe, and the viscosity of the fluid. Indeed, we should expect it to increase with the former, and to decrease with the latter two. The concrete expression which holds in this case is called the *law of Hagen and Poiseuille*:

$$G_V = \frac{\pi r^4}{8\eta l} \tag{1.27}$$

Here, η is the viscosity, and r and l are the radius and the length of the pipe, respectively. A derivation of this relation will be given in Chapter 29. The relation of Hagen and Poiseuille shows that a fluid with a viscosity equal to zero would be ideal.

Turbulent Flow

In the flow of fluids through pipes, turbulence occurs if a certain combination of fluid speed, pipe diameter, and viscosity of the fluid surpasses a critical value. For a given pipe and fluid, therefore, the speed of flow is the decisive factor. For turbulent flow, and for the transition from laminar to turbulent conditions, circumstances are much more complex. For one, the relation between flux and pressure drop now also depends upon the roughness of the pipe. We will not go into details here which are of great interest in engineering. In general, relations are presented in graphical or tabular form.

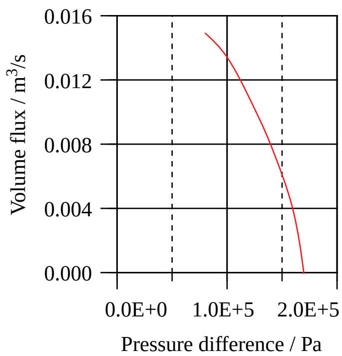
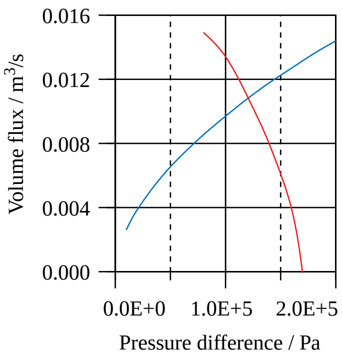


FIGURE 1.30. Characteristic curve of a pump.

Q

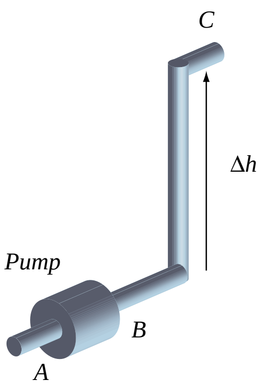
1. Devices such as pumps also have $I_V - \Delta P$ characteristic curves: all other factors kept constant, the flow depends upon the pressure difference across the device. Would it make sense to introduce a resistance to describe the phenomena associated with such devices?
2. A pump is fitted to the smooth pipe described in Fig.1.29. Water is to be pumped horizontally through it. The pump has the $I_V - \Delta P$ characteristic curve shown in the upper graph of Fig.1.30. What values will the pressure differences and flux in the system take?
3. In laminar flow, would two identical pipes in parallel or one pipe with the cross section of two have the smaller resistance?



A

1. No, it does not make any sense. A resistance is introduced to describe the effects of friction upon the flow, an effect which leads to the production of heat. These phenomena have nothing to do with what happens in a pump of in a turbine.
2. The pressure difference across the pump must be equal in magnitude to the pressure drop in the pipe. We can find the answer to the question by superimposing the curves of the flow and the pump. The values will be those of the intersection of the curves (0.011 m^3/s and $1.2{\cdot}10^5$ Pa, respectively).
3. According to the law of Hagen and Poiseuille in Eq.(1.27), doubling the cross section would lead to a reduction of the resistance by a factor of four. On the other hand, two equal pipes in parallel would only have half the resistance of a single one.

EXAMPLE 1.4. Pumping water through a smooth pipe.

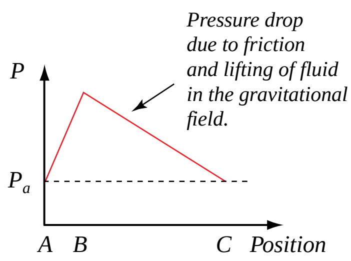


Water is pumped 5 meters upward through the rough pipe with a length of 10 m and a radius of 2 cm described in Fig.1.29. We would like to have a flux of 8.0 liters/s. a) What is the pressure difference the pump should build up for steady flow? b) Calculate the resistance and conductance. c) How large is the average speed of flow for a cross section of the pipe? d) If the flow was still laminar, would the flux be higher or lower for the given pressure difference?

SOLUTION: a) First we use the loop rule as in Eq.(1.23) for a closed loop from *A* to *B* and *C*, and back to *A* via the air. The pressure of the fluid increases from *A* to *B* (this is a value we are looking for), and it decreases back to P_a from *B* to *C*. The pressure drop is due to two processes: first, we have fluid resistance, second, the fluid is lifted in the gravitational field:

$$\Delta P_P + \Delta P_R + \Delta P_h = 0$$

where ΔP_P and ΔP_h are the pressure changes due to the pump and the vertical rise, respectively. For ΔP_R we look in the diagram for the flow characteristic curve (middle curve in Fig.1.29). A flux of 8.0 liters/s corresponds to 0.0080 m^3/s which yields a value of – $1.5{\cdot}10^5$ Pa for the pressure drop as a result of friction. For ΔP_h we have



$$\Delta P_h = -\rho g |\Delta h| = -1000\,\mathrm{kg/m^3} \cdot 9.81\,\mathrm{Pa \cdot m^2/kg} \cdot 5.0\mathrm{m} = -4.9 \cdot 10^4\,\mathrm{Pa}$$

Therefore, the pump must raise the pressure of the fluid by $1.99{\cdot}10^5$ Pa = 1.99 bar.

b) Conductance and resistance are calculated according to the conductance or resistance laws in Eqs.(1.24) or (1.25):

$$G_V = \frac{I_V}{\Delta P_R} = \frac{0.0080\,\mathrm{m^3/s}}{1.5 \cdot 10^5\,\mathrm{Pa}} = 5.33 \cdot 10^{-8}\,\frac{\mathrm{m^3}}{\mathrm{Pa \cdot s}} \quad \Rightarrow \quad R_V = 1.88 \cdot 10^7\,\frac{\mathrm{Pa \cdot s}}{\mathrm{m^3}}$$

c) Eq.(1.26) yields

$$\bar{v} = \frac{I_V}{A} = \frac{I_V}{\pi r^2} = \frac{0.0080}{\pi \cdot 0.020^2}\,\mathrm{m/s} = 6.4\,\mathrm{m/s}$$

d) For laminar flow we calculate the conductance (or the resistance) according to Eq.(1.27). From Eq.(1.24) we obtain

$$I_V = \frac{\pi r^4}{8\eta l}\Delta P_R = \frac{\pi \cdot 0.020^4}{8 \cdot 0.0010 \cdot 10} 1.5 \cdot 10^5\,\mathrm{m^3/s} = 0.94\,\mathrm{m^3/s}$$

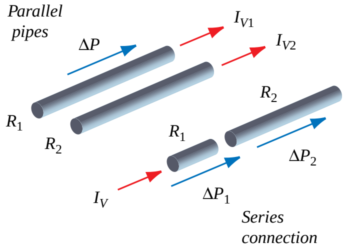
The viscosity was taken from Table 1.2. The value of the flux is about 100 times larger than the actual one which is the consequence of underestimating the fluid resistance by a factor of 100.

■

EXAMPLE 1.5. Parallel and series connections of pipes for laminar flow.

Derive the relations for the hydraulic resistance of two parallel pipes, or two pipes in series, for laminar flow. Neglect the influence upon flow of bends and connectors in the pipe systems.

SOLUTION: For parallel pipes, ΔP_R is the same for both of them, while the current through both is the sum of the single currents. Therefore:



$$R_V = \frac{\Delta P_R}{I_V} = \frac{\Delta P_R}{I_{V1} + I_{V2}} = \left(\frac{I_{V1} + I_{V2}}{\Delta P_R}\right)^{-1} = \left(\frac{I_{V1}}{\Delta P_R} + \frac{I_{V2}}{\Delta P_R}\right)^{-1} = \left(\frac{1}{R_1} + \frac{1}{R_2}\right)^{-1}$$

To obtain the equivalent resistance for parallel pipes we have to add the inverse of the single resistances and then calculate the inverse.

For pipes connected in series, the calculation proceeds analogously. Note that here the current is the same for both, while the total pressure drop is the sum of the single values:

$$R_V = \frac{\Delta P_R}{I_V} = \frac{\Delta P_{R1} + \Delta P_{R2}}{I_V} = \frac{\Delta P_{R1}}{I_V} + \frac{\Delta P_{R2}}{I_V} = R_{V1} + R_{V2}$$

The total resistance is the sum of the resistances of the single pipes. Note that the results hold in equivalent form for more than two pipes.

■

1.5 The Hydraulic Capacitance

When we built the model for the dynamics of the two communicating lakes we saw that we needed to relate pressure differences to the amount of water stored in the lakes. This is an example of the more general problem of how a level quantity changes with the amount stored in a system. Solving this problem will conclude our first investigation of fluid flow. Further issues of hydraulics will be discussed in Chapters 3 and 4.

Volume and Pressure in Fluid Containers

Containers for fluids are simple hydraulic devices which display the relationship just mentioned. Pressure vessels are another type of system where amounts of fluids stored and the pressure of the fluids are related. Let us first consider the case of the open fluid containers in Fig.1.31.

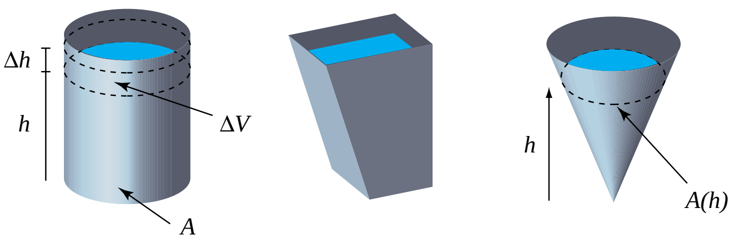


FIGURE 1.31. For a given fluid, the pressure at the bottom of a container only depends upon the height of the fluid column. For a given container this quantity is related to the amount of fluid stored.

We need to know the fluid pressure at the bottom of a container. The problem is simple to solve for straight walled tanks (the leftmost container in Fig.1.31). There we have

$$P_h = \frac{\rho g}{A} V \qquad \textbf{(1.28)}$$

if we neglect the pressure at the surface of the fluid. For more general shapes of tanks the problem is cast in a different form. We consider how much the pressure of the fluid column changes if the volume is changed. The relation between the two quantities depends upon the particular container. If the tank is wide at the current level of fluid, it takes more fluid for a certain change of pressure. If it is narrow, it takes less. There is a quantity pertaining to the tank and the fluid which relates changes of pressure to changes of volume:

- The quantity which tells us—for a given container—how fast the volume of fluid stored must change if we want to have a certain rate of change of the pressure at the bottom, is called the *hydraulic capacitance* K_V:

$$dV/dt = K_V dP/dt \tag{1.29}$$

P_C is the pressure related to the capacitance. In general, K_V is variable: it depends upon the cross section of the tank. For relatively small changes for which K_V can be considered constant we can write Eq.(1.29) in the form

$$\Delta V = K_V \Delta P_C \tag{1.30}$$

where ΔP_C is the change of pressure as the consequence of amounts of fluid stored in a container. Eq.(1.29) tells us that—for the straight walled tank—the hydraulic capacitance must be the factor $A/(\rho g)$, and it is constant. For the general case, we obtain essentially the same answer, only with A replaced by the variable cross section $A(h)$:

$$K_V = \frac{A(h)}{\rho g} \tag{1.31}$$

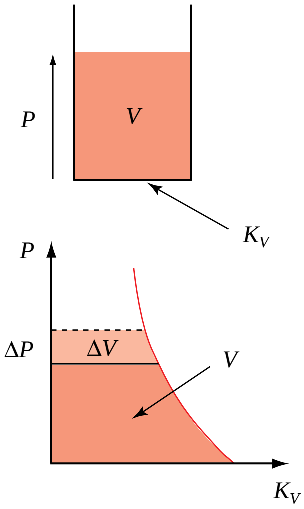


FIGURE 1.32. A graph of P versus K_V—which resembles the shape of the container—is used to determine volume as a function of pressure, or volume changes as a function of changes of pressure. The upper diagram is the source of this image: in the case of a tank with straight walls, the content of the symbolic container represents the volume, while the bottom and the fluid level represent the capacitance and the pressure of the fluid, respectively.

The Graphical Meaning of Capacitance

There is a simple graphical procedure of relating the volume stored to the pressure of the fluid column which derives directly from the image of the fluid container (Fig.1.32). If we draw the pressure as a function of capacitance, we get a graph resembling the shape of the container. Remember that the pressure is proportional to the height of the fluid in the tank, while the capacitance is proportional to the cross section at a given height (i.e., pressure). The volume of fluid is read from the graph as the area between the curve $P(K_V)$ and the P–axis up to pressure P. Changes in volume are equal to the area of small rectangles of width ΔP and height K_V, which we can see by interpreting Eq.(1.30) graphically.

General Hydraulic Capacitance

Finally, let us consider a different kind of fluid storage which nevertheless still displays the same general relationship between storage and pressure, and which, therefore, is described also in terms of a capacitance. Take a pressure vessel for liquids with a deformable membrane. The simplest version of this container is a small rubber balloon filled with water. Adding more water will increase the pressure of the liquid, and there exists a general relation of the form of Eq.(1.29). The characteristic curve of the rubber membrane will determine the capacitance as a function of pressure for this container. We can even extend this example to pressure vessels for gases. Usually, however, we work with constant volumes as we press more gas into the tank. In this case we should express the capacitance in terms of changes of the amount of gas, rather than the volume, and changes of pressure.

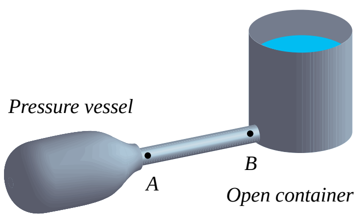
Q

1. What is the meaning of the hydraulic capacitance? Explain the difference for containers storing liquid vertically, and for pressure vessels for liquids and gases.
2. What is the shape of the liquid container having a capacitance such as the one in Fig.1.32? What is the meaning of the horizontal distance between the P–axis and the P–K_V curve? What is this quantity proportional to?
3. Does the general definition of capacitance relate volume to pressure, or changes of volume to changes of pressure? Does capacitance mean how much a system can store?

A

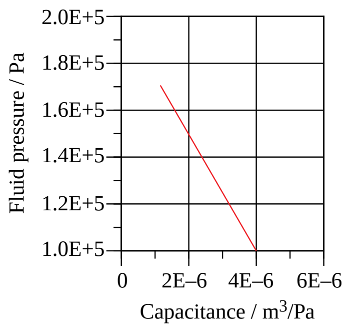
1. Hydraulic capacitance is the quantity which relates the rate of change of pressure to how fast the volume changes. This form of the relation is valid also for pressure vessels. In the case of gases stored at constant volume we should replace the volume of the fluid by amount of gas.
2. The container gets narrower at the top. The distance in the graph is the capacitance at a given pressure (of the liquid at the bottom). It is proportional to the cross section of the tank at the given height.
3. It relates changes, rather than the values themselves. No, it does not.

EXAMPLE 1.6. Fluid vessels and flow.



Consider a pressure vessel connected to an open, straight-walled container by a pipe. The container is open to the air which has a pressure of 1.0 bar. The system contains oil with a density of 800 kg/m^3 and a viscosity of 0.20 Pa·s. The pressure-capacitance relation for the pressure vessel is given in the accompanying graph. The cross section of the tank is 0.080 m^2. The length and the radius of the pipe measure 2.0 m and 0.020 m, respectively. a) At a certain point in time the pressure vessel contains 0.15 m^3 of oil, whereas the tank has stored 0.20 m^3. Calculate the flux of oil through the pipe. In which direction does the oil flow? b) What will the final level of oil in the container be?

SOLUTION: a) We first calculate the pressure of the fluid in the vessel from the pressure-capacitance curve. Then we determine the level and the pressure of the oil in the tank. The difference of pressures lets us calculate the flux if we know the hydraulic resistance for the flow through the pipe. The latter will be given by the law of Hagen and Poiseuille.



The lowest possible pressure of the oil in the pressure vessel is 10^5 Pa. Therefore, we calculate the volume of fluid in the container by the area between the pressure-capacitance curve and the vertical axis starting at $P_a = 10^5$ Pa. See Fig.1.32. Since the capacitance curve is a linear function, namely

$$K_V(P) = K_V(P_a) - a(P - P_a) \quad , \quad K_V(P_a) = 4.0 \cdot 10^{-6}\,\mathrm{m^3/Pa} \quad , \quad a = 4.0 \cdot 10^{-11}\,\mathrm{m^3/Pa^2}$$

the calculation is simple (you can read it from the graph):

$$V(P_v) = \frac{1}{2}\left(K_V(P_a) + K_V(P_v)\right)(P_v - P_a) \quad \Rightarrow \quad P_v = 1.50 \cdot 10^5\,\mathrm{Pa}$$

Here, P_v denotes the pressure of the liquid in the pressure vessel. The pressure of the fluid at the bottom of the tank, on the other hand, may be calculated with the help of the capacitance of this container:

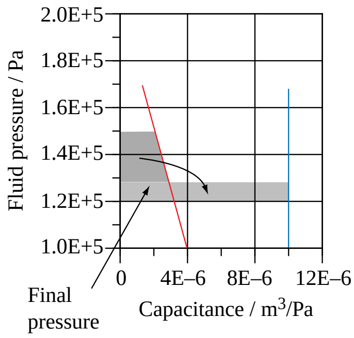
$$K_V = A/(\rho g) = 0.080/(800 \cdot 10)\,\mathrm{m^3/Pa} = 1.0 \cdot 10^{-5}\,\mathrm{m^3/Pa} \quad \Rightarrow \quad P_t = 1.20 \cdot 10^5\,\mathrm{Pa}$$

Obviously, the oil must flow from the pressure vessel to the tank. To compute the flux of oil we need the hydraulic resistance:

$$R_V = \frac{8\eta l}{\pi r^4} = \frac{8 \cdot 0.20 \cdot 2.0}{\pi 0.020^4} \frac{\text{Pa} \cdot \text{s}}{\text{m}^3} = 0.64 \cdot 10^7 \frac{\text{Pa} \cdot \text{s}}{\text{m}^3}$$

from which we conclude that

$$|I_V| = \frac{|\Delta P_{A-B}|}{R_V} = \frac{0.30 \cdot 10^5}{0.64 \cdot 10^7} \frac{\text{m}^3}{\text{s}} = 0.47 \cdot 10^{-2} \frac{\text{m}^3}{\text{s}}$$



b) Oil will flow from the vessel to the tank until the fluid pressures in both of them have become equal. The outcome of this process can be represented in the pressure-capacitance diagram. Since the changes of volume associated with changes of pressure are shown as areas, we can see that the final pressure attained will make the absolute values of the volume changes equal for the containers (balance of volume):

$$\Delta V_v = -\Delta V_t$$

The index t denotes the tank. These changes can now be expressed in terms of the changes of pressure:

$$\frac{1}{2}\left(K_{Vv}\left(P_{vi}\right) + K_{Vv}\left(P_{vf}\right)\right)\left(P_{vi} - P_f\right) = -K_{Vt}\left(P_{ti} - P_f\right)$$

where P_f is the final pressure attained. Solution of the equation leads to P_f = 126 kPa, leading to a final value of 0.260 m^3 of oil in the tank.

■

1.6 Diagrams for Modeling Hydraulic Processes

Diagrams are an important visual tool which helps in creating the abstractions necessary for modeling of dynamical processes. Remember our basic image of how nature works: flow, production, and storage of certain easily visualized quantities are the cause of what we observe. Moreover, the quantities flow through differences of levels: “downhill” for voluntary processes, “uphill” for involuntary ones.

Here we will study two types of diagrams helpful in constructing models. The first we call a *system diagram*, while the second—which we have already encountered in the Introduction and this chapter—is the well known *system dynamics diagram*.

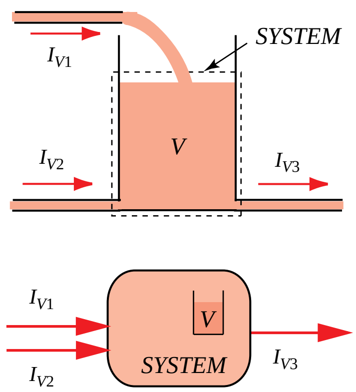


FIGURE 1.33. Symbolic representation of flow and storage in a system diagram. The diagram suggests a law of balance. This law, however, is better represented by a system dynamics diagram.

System Diagrams

The goal is to represent the image underlying natural processes with the help of diagrams. Since flow, storage, and levels are the main concepts we deal with, we need a tool for visualizing these. Here, we construct simple system diagrams for some of the important hydraulic phenomena.

First, consider a fluid container with inlets and outlets. We are concerned with flow, storage, and balance. A diagram for representing flow and storage—but not balance itself— starts with a box. Flows are visualized as arrows leading into or out of the box, while storage is symbolized by a little container inside the box (Fig.1.33).

In a second step, we add the information concerning hydraulic levels, i.e., the pressure of the fluid as it flows into or out of the system. The values of the pressure are those of the fluid at the system boundary. Inside the system, the pressure may vary from point to point. Flows and levels permit us to draw system diagrams of such fundamental hydraulic systems as turbines and pumps (Fig.1.35), and pipes (Fig.1.36).

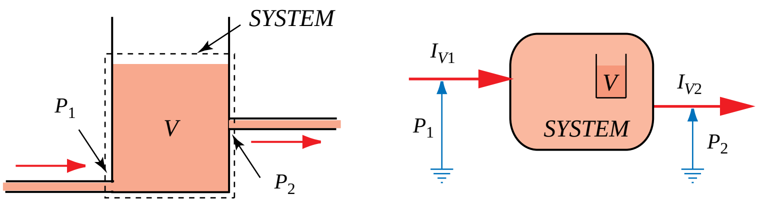


FIGURE 1.34. Adding the information concerning hydraulic levels to the system diagram. The hydraulic level is measured by the pressure of the fluid.

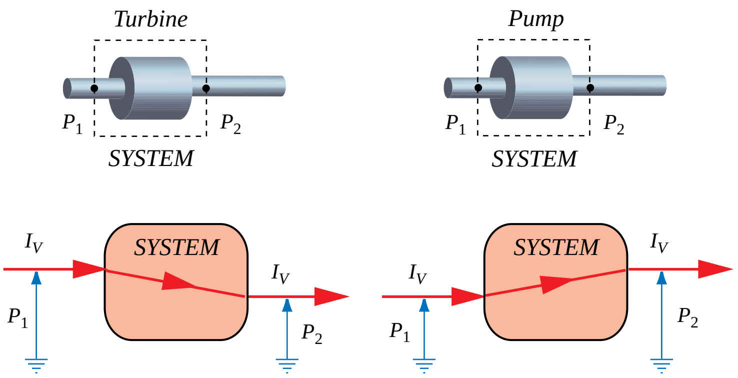


FIGURE 1.35. System diagrams of a turbine and a pump (from left to right). The system diagrams show flows and levels, and they indicate whether the flow goes up or down with respect to the hydraulic level. Flows going down are said to be voluntary hydraulic phenomena: they drive other processes. Flows going up are involuntary: they must be driven.

Note that system diagrams are snapshots: they do not show time evolution; rather, they represent conditions at a fixed moment at various points in a system or combined system. In general, they are not directly useful for calculations. Their strength lies in presenting in a simple way the most important processes. They are a tool for talking coherently about the problems posed—guiding us in the use of the systems view of physical processes. They can be used as steps toward more formal representations of what we see happening. The systems diagrams will be particularly useful when we include the discussion of energy with the description of the processes (Chapter 4).

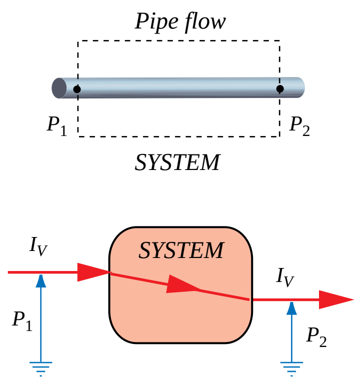


FIGURE 1.36. System diagram of fluid flow with resistive characteristic through a pipe. The fluid flows from a point of high to a point of low pressure.

System Dynamics Diagrams

If we wish to model the dynamical aspects of processes with the goal of calculating system behavior, we have to turn to a tool which permits us to represent the relations underlying nature. Hydraulics tells us what these relations are:

1. *Laws of balance* which relate the fluxes to the rate of change of system content. They serve as the backbone of any formulation of models of dynamical processes.
2. *Capacitance laws* which express fluid pressure in terms of the volume of fluid stored.
3. *Resistance laws* which relate flows to pressure drops, and the concrete relations for a particular type of flow.
4. The *loop rule* which states that pressure differences along a closed loop add up to zero.

System dynamics tools provide a small collection of building blocks which is all we need to formulate our view of the dynamics of physical processes. This we have already seen by way of example in the Introduction and in Section 1.1.

There are a few fundamental substructures of a model of hydraulic systems from which complete—and complex—models can be built (Fig.1.37). These elements represent the four relations listed above, i.e., the law of balance, the laws of capacitance and resistance, and the loop rule.

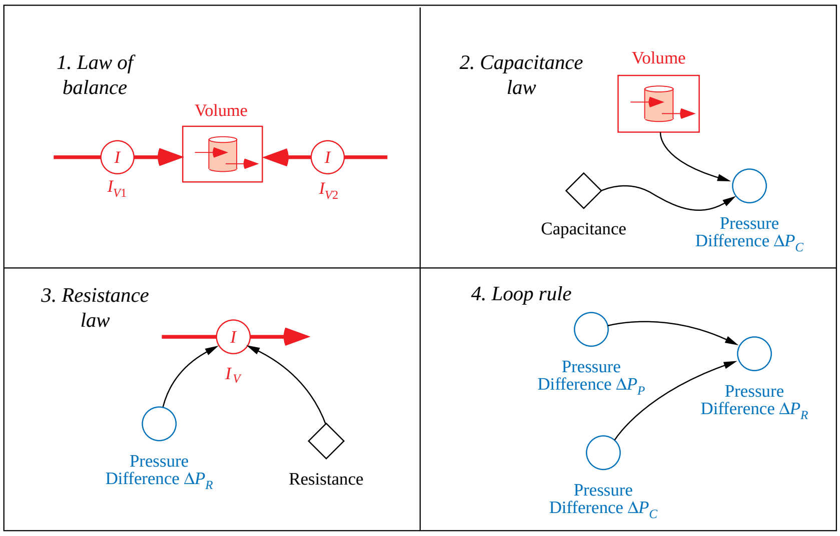


FIGURE 1.37. Four basic substructures of system dynamics models of hydraulic processes: the laws of balance, capacitance, resistance, and the loop rule.

System Dynamics Models of Hydraulic Processes

To demonstrate the use of these consider the example of a viscous oil flowing out of a tank with a horizontal pipe fitted at the bottom, such as in Fig.1.24. All four substructures are needed, and the two parameters—capacitance and resistance—can be calculated from Eqs.(1.31) and (1.27). The system dynamics diagram is shown in Fig.1.38.

$$\dot{V} = I_V$$
$$V(t=0) = V_o$$
$$I_V = \Delta P_R / R_V$$
$$\Delta P_R = -\Delta P_C$$
$$\Delta P_C = V / K_V$$

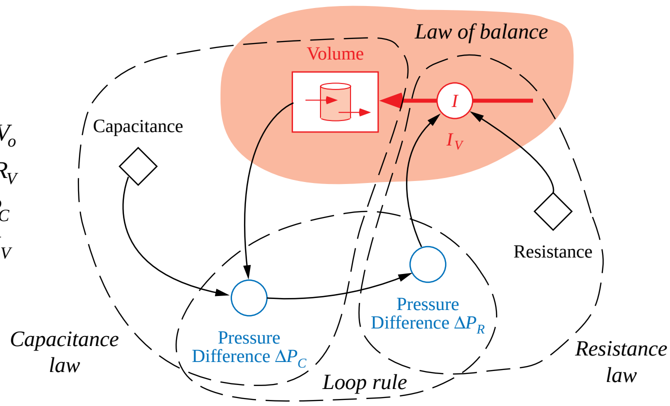
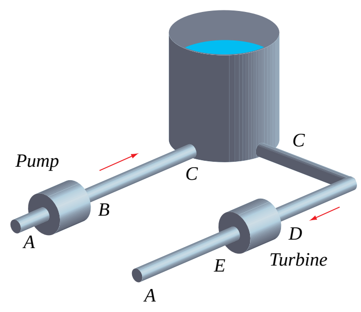


FIGURE 1.38. System dynamics diagram of a simple hydraulic problem. The diagram represents the process of draining of a tank with straight walls through a horizontal pipe fitted at the bottom (Fig.1.24). The flux–pressure characteristic curve is linear as in the law of Hagen and Poiseuille.

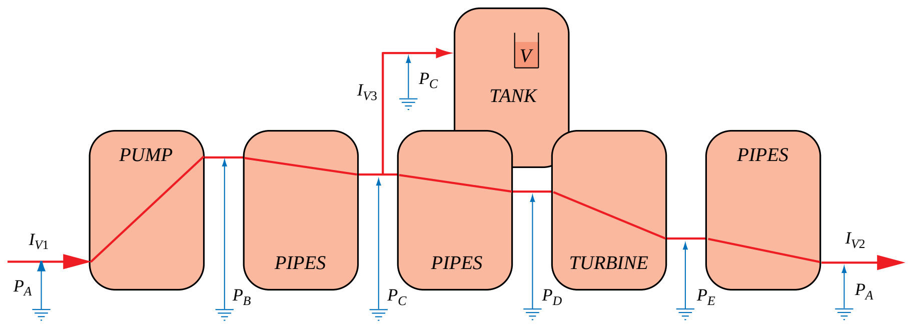
A system such as the one modeled in Fig.1.38 is called an *RC system* in analogy to systems made up of resistors and capacitors in electricity. *R* stands for resistance, while *C* denotes the capacitance. In Chapter 2 we will demonstrate how we can make use of the analogies which are suggested by our systems view of nature. Chapter 3, finally, will introduce us to another interesting system property—namely inductance.

EXAMPLE 1.7. Hydraulic circuit diagram.



Consider a hydraulic circuit consisting of a pump, turbine, pipes, and fluid storage as in the accompanying figure. Consider the fluid to be taken from and discharged to a large shallow pond at ground level. Draw a flow diagram combining the system diagrams for each of the important components.

SOLUTION: Let us begin with the pump. The pump, pipes leading to the storage tank, pipes leading to the turbine, the turbine, and the pipes leading back to the pump form a closed circuit: the elements are connected in series. The pond closes this loop. The pressure of the fluid is raised in the pump. From there, the pressure decreases through the pipes and through the turbine.



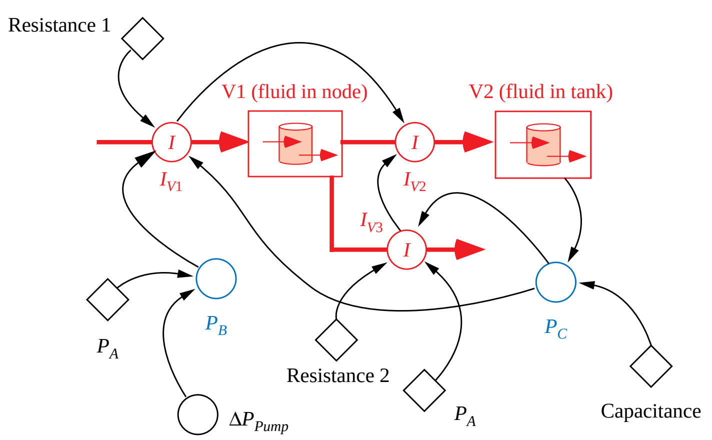
The tank is connected in parallel to the circuit section *A-B-C*. In the fluid column, the pressure decreases from P_C to P_A. Note that the fluid current splits at point *C*. The sum of the three fluxes at *C* is equal to zero.

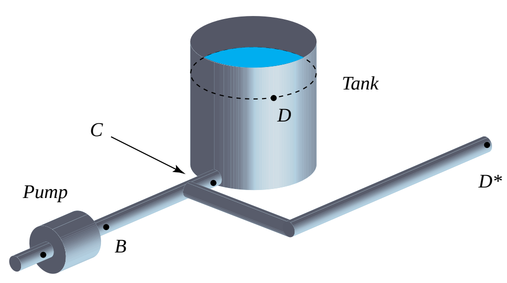
■

EXAMPLE 1.8. System dynamic model for filling a tank.

The pump in the system shown in Fig.1.27 is operated at constant pressure as the tank is filled. The flow obeys the law of Hagen and Poiseuille. Assume the pipe from the pump to the tank and the pipe leading away from the tank to be identical. a) Represent the processes with the help of a system dynamics model diagram. b) To what height can the tank be filled?

SOLUTION: a) The main features of the model are these. First, we represent the law of balance of volume for the tank by a stock and a single flow (I_{V2} from point *C* in Fig.1.27 to the tank). This current comes from the node at point *C* which is also represented by a stock.





Since the node cannot store fluid, the three fluxes associated with point C satisfy the condition that I_{V1} equals the sum of I_{V2} and I_{V3}. I_{V1} and I_{V3} can be calculated from the resistance law if the pressure differences across the pipes from B to C, and from C to D^* are known. All we have to do now is calculate P_B from knowing the pressure difference set up by the pump, and P_C from the capacitance law for the tank.

b) The maximum height to which the tank can be filled depends upon the maximum pressure difference available for the fluid column from C to D. Since the tank and the second pipe are connected in parallel, the pressure difference is the same as the one from C to D^*. The latter, however, depends upon the current of fluid flowing through this pipe.Now, the current through the second pipe must be equal to that through the pump and the first pipe; the reason is simply that the fluid is not flowing any longer into the tank when the maximum height of the fluid column has been reached. Therefore, the fluid directly flows from the pump through C and out at D^*. Since the pipes are identical, each of them takes half the pressure difference set up by the pump. Obviously, we now have

$$\Delta P_{CD^*} = \frac{1}{2}\Delta P_{pump} \quad , \quad \Delta P_{CD} = \Delta P_{CD^*} \quad , \quad \Delta P_{CD} = \rho g \Delta h_{max}$$

from which we conclude that $\Delta h_{max} = \Delta P_{pump}/(2\rho g)$.

■

Chapter Summary

We have studied a small aspect of the general field of fluid flow which is called hydraulics. Hydraulics studies fluids in controlled environments—pipes and containers—often for the purpose of technical applications. Despite its limitations, it tells us much about the behavior of nature. The most important points to remember about hydraulic processes are these:

- Hydraulic processes have to do with the flow and the storage of amounts of fluid. If we consider only incompressible fluids such as water or oil, we can use their volume as an easy measure of their amount.
- The fundamental behavior of fluids is expressed by the law of balance of volume. With incompressible fluids—and disregarding chemical reactions—the volume of fluid in a system can only change as the result of inflow and outflow (Section 1.2). Therefore, the sum of all volume fluxes tells us how fast the volume of the fluid stored must change.
- If a fluid is at rest in communicating containers, the levels will be the same, not the volumes. Level differences—and with them pressure differences—can be the cause of flows. In a closed hydraulic circuit, the sum of all pressure differences must be zero (loop rule; see Section 1.3).
- Normally, there is friction in fluid flow (as a consequence of viscosity). Therefore, the pressure of the fluid must decrease in the direction of flow. We need a pressure difference to force a fluid through a pipe, and this pressure difference is related to the flow by a resistance law (Section 1.4): the flow equals the pressure difference divided by the hydraulic resistance.
- In fluid containers, the pressure of the fluid is related to the amount of fluid stored by a capacitance law (Section 1.5): the rate of change of pressure multiplied by the capacitance tells us how fast the volume stored must change.
- Combining laws of balance with laws for flows and containers, and with the loop rule, leads to complete models of dynamic hydraulic processes.

Questions

1. Which phenomena are related by a law of balance? What is the law of balance of amounts of water?
2. Are you sure that flows of water are the only processes which can change the amount of water in a system? If they are not, what law would have to be changed, and how?
3. Consider Fig.1.8. Why is there a minimum of the volume of water stored in the system at about $1.1 \cdot 10^5$ s? What can you say about the fluxes of water before this point in time? After this point in time?
4. Why does the strength of the gravitational field play a role in the pressure of a fluid column? Why does the pressure increase linearly with depth if the fluid is a liquid such as water?
5. Consider two communicating lakes with a single channel connecting them, and without further inflows or outflows. Is the system at rest when there are equal amounts of water in the lakes?
6. What is the relation between viscosity, friction, and production of heat in fluid flow? What are the conditions of ideal flow?
7. In Example 1.4 we calculated the pressure difference to be set up by a pump for pumping a fluid at steady state. Does your answer depend on whether the fluid entering the pump already flows or is pumped from rest?
8. In Fig.1.26, a straight line connecting the exit point of the pipe and the top level of the glass tubes showing the pressure of the fluid goes to the top of the fluid layer in the tank (upper figure in Fig.1.39). Which model assumptions make this to be the case? Actually, it is found that the line goes to a point well below the fluid surface (lower figure). What is the reason for this behavior? What did we neglect?

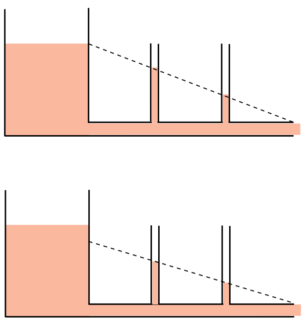


FIGURE 1.39. Question 8.

9. We have associated the notion of resistance and capacitance with flow through pipes, and with containers storing fluids, respectively. Couldn't a capacitor also be a resistor, and vice-versa?
10. What does the characteristic curve of a pump look like which sets up a constant pressure difference independent of the flux of fluid?
11. Use the abstract "hydraulic" image of a straight walled tank containing a fluid to explain the relation between content, level, and capacitance. Which geometrical quantity corresponds to which physical one?
12. Explain the meaning of the terms "voluntary" and "involuntary" hydraulic processes. In what sense is the phenomenon of resistive flow of a viscous fluid through a pipe a voluntary hydraulic process? How is this expressed in a system diagram?
13. What is the system diagram of a fluid discharging from a tank such as in Fig.1.24?
14. In Fig.1.38, the equation of balance (the first in the list of equations) does not have to be written explicitly in software tools. Why is this the case?
15. What is the behavior over time of the currents through the two pipes in the system of Example 1.8? Use qualitative reasoning.

Exercises

1. A pump forces water through a long pipe, and then through a turbine. Draw the combination of system diagrams representing this system.
2. Calculate the pressure of the water at the bottom of a lake 100 m deep.
3. A person's blood pressure is said to correspond to 130 mm of a column of mercury. Determine the pressure.
4. Castor oil is pumped through a pipe 10 m long, having a diameter of 5.0 cm. Determine the hydraulic resistance.
5. Water is pumped through a smooth pipe having, respectively, a length of 10 m and a diameter of 4.0 cm. The flux is 10 liters/s. The water then flows through a turbine, and back to the pump through another pipe having the same dimensions as the first. The pump sets up a pressure difference of 5.0 bar. What is the pressure difference across the turbine?
6. Determine the hydraulic capacitance of a swimming pool 25 m long and 15 m wide.
7. Two communicating straight-walled containers having diameters of 0.40 m and 0.60 m, are filled with olive oil to levels of 1.0 m and 0.30 m, respectively. What is the common final height of the oil in the tanks?
8. In the process modeled in Fig.1.38, the volume of fluid in the tank as a function of time takes a form similar to the one shown in the graph of Fig.1.40. Why is this so? How large is the (negative) slope of the curve right at the beginning?

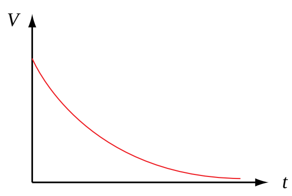


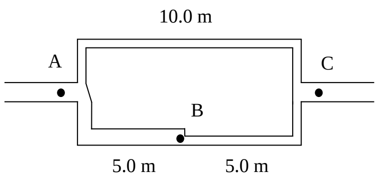
FIGURE 1.40. Exercise 8.

Problems

1. Two currents of water are flowing into a fountain. The first changes linearly from 2.0 liters/s to 1.0 liters/s within the first 10 s. The second has a constant magnitude of 0.50 liters/s. In the time span from the beginning of the 4th second to the end of the 6th second, the volume of the water in the fountain decreases by 0.030 m^3. a) Calculate the volume flux of the current leaving the fountain. b) How much water will be in the fountain after 10 s, if the initial volume is equal to 200 liters?

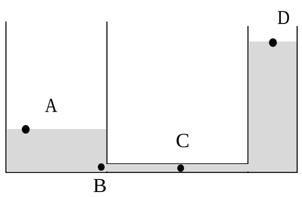
2. Oil having a density of 800 kg/m^3 and a viscosity of 0.60 Pa·s is flowing through a system of pipes as in. The pressure at point A is 1.40 bar, while at point C it is 1.20 bar. The thin pipes have a diameter of 1.0 cm while the diameter of the piece leading from A to B has a diameter of 2.0 cm. Neglect the influence of corners in the pipes, and assume the law of Hagen and Poiseuille to apply. a) What is the value of the volume flux through the lower pipes? b) How large is the total volume flux through the pipes? c) What is the pressure at B?

FIGURE 1.41.
Problem 2.



3. Two tanks (see Fig.1.42) contain oil having a density of 800 kg/m^3 and a viscosity of 0.20 Pa·s. Initially, in the container at the left, which has a cross section of 0.010 m^2, the fluid stands at a level of 10 cm; in the second container (cross section 0.0025 m^2) the level is 60 cm. The hose connecting the tanks has a length of 1.0 m and a diameter of 1.0 cm. a) Calculate the pressure at A, B, C, and D at this point in time. The pressure of the air is equal to 1.0 bar, and C is in the middle of the hose. b) Sketch a pressure profile (pressure as a function of position) for a path leading from A to D; include a point C* at the other end of the pipe from point B. c) What is the volume current right after the hose has been opened?

FIGURE 1.42.
Problem 3.



4. A large and shallow lake is going to be filled through a horizontal pipe with a length of 10 km. Initially the lake is empty; in the end it is supposed to contain 10^5 m^3 of water. Assume the hydraulic resistance to be modeled by the law of Hagen and Poiseuille; i.e., take the volume flux to be proportional to the pressure difference across the pipe. The pressure drops by 10^2 Pa per meter of length at a volume flux of 1.0 m^3/s. While the lake is being filled, water evaporates from its surface at a rate of 0.10 m^3/s. a) If the volume flux is constant and equal to 0.50 m^3/s, what is the rate of change of the volume of water in the lake? b) How long will it take to fill the lake? c) How large is the pressure difference set up by the pump?

5. A large oil tank is filled through a pipe at its bottom (as in Fig.1.43). The flow of oil through the pipe is supposed to be laminar. Derive the instantaneous volume flux in terms of the length and the radius of the pipe, the viscosity and density of the oil, and the height of the oil in the tank.

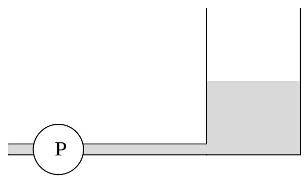


FIGURE 1.43.
Problem 5.

6. Two tanks (see Fig.1.42) contain oil with a density of 800 kg/m^3 and a viscosity of 0.20 Pa·s. Initially, in the container, which has a cross section of 0.010 m^2, the fluid stands at a level of 10 cm; in the second container (cross section 0.0025 m^2) the level is 60 cm. The hose connecting the tanks has a length of 1.0 m and a diameter of 1.0 cm. Sketch the levels in the containers as a function of time.

7. Calculate the hydraulic capacitance of a U-shaped glass tube used in a mercury pressure gauge. The inner diameter of the tube is 8.0 mm.

8. Calculate the hydraulic capacitance of a conically shaped fluid container as a function of fluid pressure at the bottom (see Fig.1.31).

9. Two containers are joined by a pipe as in. The second container has both an inlet and an outlet. Assume the flow through the pipes to obey the law of Hagen and Poiseuille. a) Write the equations of balance of volume for the fluid in the containers. b) Derive the relation between volume of fluid and pressure of fluid at the bottom of each of the containers. c) Write the laws for the volume fluxes through both pipes. d) Derive the differential equations for the height of the fluid in each of the containers in terms of the hydraulic capacitance and resistance of the elements of the system.

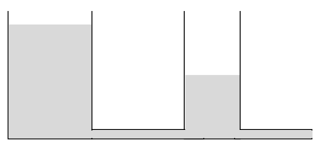


FIGURE 1.44.
Problem 9.

10. For the system of container, pipes, and pump shown in Fig.1.27 derive a) the instantaneous pressure difference across the pump, and b) the instantaneous volume flux through the pump. The fluid is oil as in Problem 6. It stands at a level of 1.0 m in the tank having a diameter of 1.0 m. The pipes have a diameter of 5.0 cm, and lengths 2.0 m (B to C) and 3.0 m (C to D*), respectively.