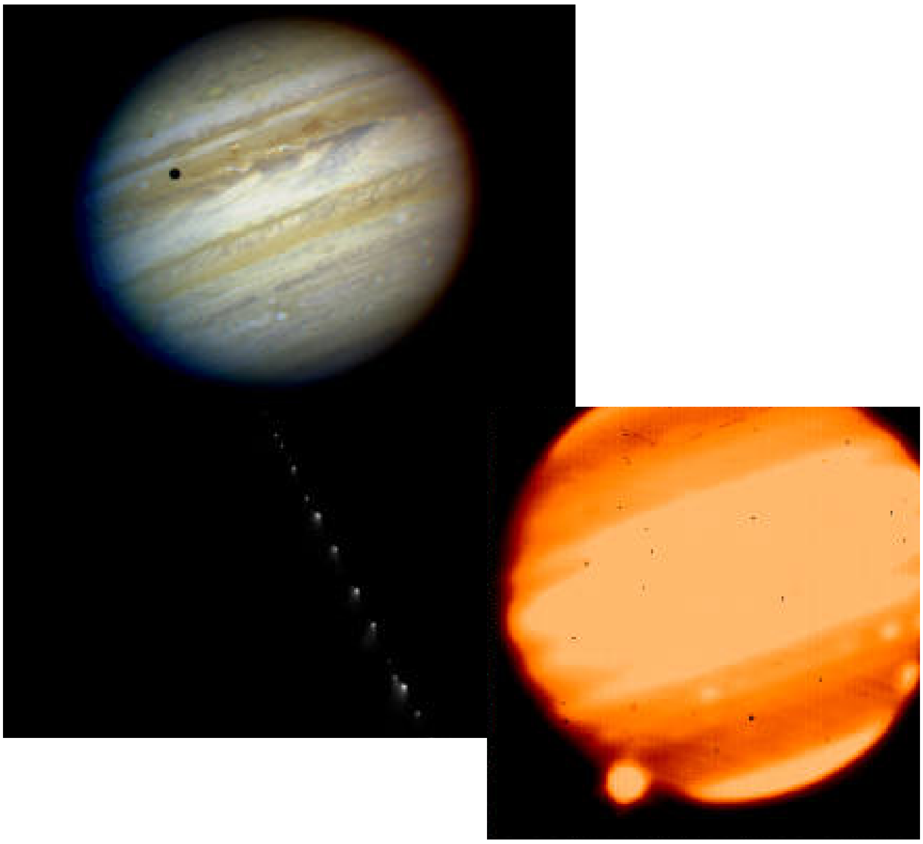
CHAPTER 13

Storage, Transfer, and Balance of Momentum



Collision of comet Shoemaker-Levi with planet Jupiter. Collisions of bodies demonstrate the properties of momentum.

CONTENTS

Mechanical processes can be explained just like electrical or thermal ones: they are the result of the storage and the transfer of a couple of fundamental quantities. We will find here that translational phenomena are the result of the storage and transport of momentum. Indeed, we say that a mechanical process is taking place if momentum (or angular momentum) is transported from one body to another. The geometrical side of motion—the fact that moving bodies change their location—does not really characterize a mechanical process (Chapter 14).

13.1 Collisions: Storage, Exchange, and Conservation of Momentum

Collisions nicely demonstrate the fundamental property of bodies in motion: they contain a quantity of motion if they move, and they transfer this quantity if they interact mechanically with other bodies. We notice this, for example, if a person runs into us, if a snow ball hits us, if a ball is hit by a bat, or if we run into a wall. A body at rest does not have this effect if we just touch it. Obviously, a moving body carries a quantity of motion, more, if it is faster, and more, if it is bigger. In collisions, quantities of motion are exchanged from body to body.

Examples of Collisions

Collisions are an important class of mechanical phenomena. They occur at every scale—large and small, fast and slow. Here are some examples:

- Steel balls on strings hitting others (Fig.13.1).
- Coins or billiard balls colliding on a table.
- Train cars when hooking up.
- Elementary particles in accelerators.
- Planets, stars, and galaxies.

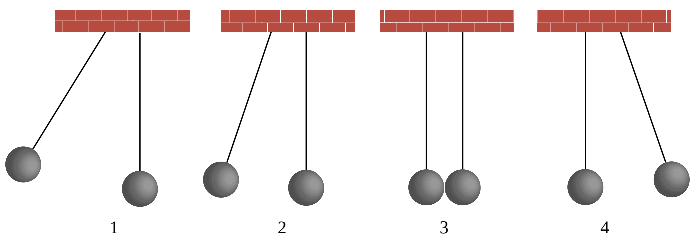


FIGURE 13.1. Identical steel balls colliding. If the first ball is released at a certain height and then collides head on with a second one hanging without moving, the latter moves off at the final speed of the first, while the first comes to rest. This is an example of a completely elastic collision.

Take a closer look at the phenomenon shown in Fig.13.1. The moving body has a certain quantity of motion just before the collision. The interaction leads to this quantity being transferred to the second steel ball. The first body must have lost all of its motion, since it comes to rest after the collision. Collisions can be anything between totally *elastic* and totally *inelastic*. The latter occurs when two bodies colliding join or merge to form a single body.

Large cover picture: Composite Hubble Space Telescope image of Jupiter and Comet Shoemaker-Levi. Small inset: Infrared image of fireball of Fragment G on July 18, 1994, Keck Observatory.

Extensive and Intensive Quantities in Translational Motion

Here is an image which tells us more about the nature of the quantity of motion of a moving body. Divide a moving body into two parts. Obviously, both parts move at the same speed: the speed is not divided among the parts (Fig.13.2). However, we imagine a body possessing a quantity of motion which is divided into two parts whose sum is equal to the quantity of motion of the entire body. The quantity of motion is commonly called *momentum*.

Each of the basic classes of phenomena treated in previous chapters demonstrated the existence of an extensive and an associated intensive quantity (Table 13.1). This is no different for translational motion. Indeed, momentum and velocity play the same roles for mechanics as do charge and electric potential for electric phenomena.

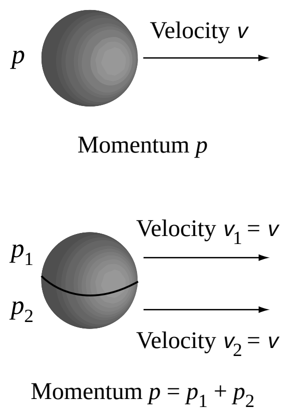


FIGURE 13.2. If a moving body is divided into two parts, each part moves at the same speed. However, each section only contains a part of the momentum of the entire body.

TABLE 13.1. A list of extensive and intensive quantities

Phenomenon	Extensive quantity	Intensive quantity
Hydraulics	Volume	Pressure
Electricity	Charge	Electric potential
Gravity	Gravitational mass	Gravitational potential
Motion (Rotation)	Angular momentum	Angular speed
Chemistry	Amount of substance	Chemical potential
Heat	Entropy	Temperature
Motion (Translation)	Momentum	Velocity

Conservation of Momentum

Mechanical phenomena here on Earth gives us the impression that motion dies by itself and that it has to be initiated again and again. By itself, a rolling ball comes to rest, and we have to push it if we want it to move again. In fact, if we ask if momentum can be created and destroyed, we have to explain a number of everyday phenomena:

- I am starting to walk; my momentum increases. A car accelerates on a street; its momentum increases (Fig.13.3a). Is momentum created?
- A sliding body comes to rest. Is momentum destroyed?
- Two identical bodies moving toward each other with equal speed. They collide and stick together. They come to rest. Is their momentum destroyed?
- Two gliders at rest on an air track have a compressed spring between them. The spring is released, and the bodies move apart (Fig.13.3b). Is momentum created in this process?

Actually, the conclusion in all cases is that momentum cannot be created or destroyed. In the first two example, momentum comes from the Earth or flows into the Earth. As a result, our planet experiences motion which is so slow that we cannot perceive it. The third and fourth example clearly demonstrate that we should think of momentum as a quantity which may be either positive or negative. In collisions, momentum is not lost. The spring pushing the gliders apart serves as an agent to separate momentum into positive and negative quantities.

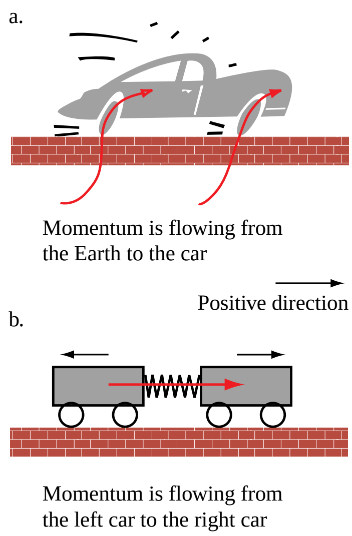


FIGURE 13.3. An accelerating car gets its momentum from the earth. Two cars being pushed apart by a spring separate momentum into positive and negative quantities. Momentum flows from the left to the right car.

Falling and Pumping of Momentum

In the examples of motion treated here, momentum flows either “uphill” or “downhill.” In other words, it flows either from a body having low velocity to one having a higher velocity, or vice-versa.

If momentum is flowing “downhill,” we speak of a *voluntary* mechanical process; if it is pumped “uphill,” the process is called *involuntary.*

Q

1. A train car is moving toward another one which is at rest. After the collision, the cars are coupled to each other. For a little while after the collision, friction is negligible. From where to where does momentum flow? Does it flow “uphill” or “downhill?” What is the change of the momentum of the first car compared to that of the second one?
2. A billiard ball is hitting a second one at rest head on. Is momentum flowing “downhill” or “uphill” in this process?

A

1. Momentum flows from the first car to the second. Since the velocity of the first car is always larger than or equal to that of the second, momentum is flowing “downhill.” The changes of momentum of cars 1 and 2 must be equal in magnitude, but of opposite sign. The change of momentum of car 1 is negative.
2. In this collision, there are two phases. First, the moving ball is slowing down until its speed is equal to that of the second. In the second phase, the speed of the first ball is still decreasing while that of the second one increases further. Therefore, during phase 1, the hitting ball is faster, in phase 2 it moves more slowly than the ball which was hit. Therefore, momentum first flows downhill before it is pumped uphill.

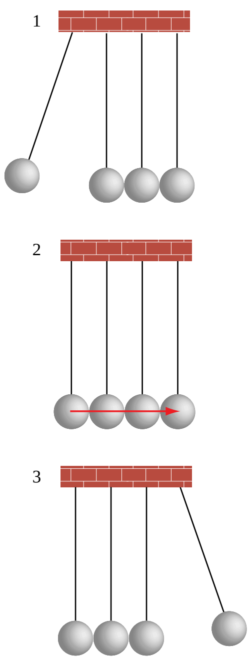


FIGURE 13.4. The left-most steel ball swings toward three balls which are at rest. If the balls are identical, and if the collision is head on, the moving sphere will come to rest, and only the right-most ball will move away at the speed at which the left sphere had just before the impact. During the collision, momentum must flow through the two balls which remain at rest.

13.2 Modes of Momentum Transfer

In collisions, bodies loose or gain momentum. In general, in mechanical processes involving translational motion. momentum is transferred. The transfer mechanisms can take three different forms.

Momentum Transfer as the Result of Bodies Touching

In collisions, when we start moving or come to rest on the ground, or if bodies are pulled or pushed—with ropes, hands, sticks, springs, and so forth—the bodies involved in the processes directly touch. Therefore, momentum must be transferred *across the surfaces* of the bodies involved, and *through the bodies* themselves. In this case we speak of *conductive transfer* of momentum. Again, the example of steel balls swinging from strings and colliding demonstrate, that bodies are capable of conducting momentum (Fig.13.4).

Conductive transfer of momentum is directly noticeable: it leads to mechanical stress in the bodies through which momentum flows. This is what we feel in mechanical interactions.

Momentum currents and stress. In mechanical processes, momentum flows. To quantify the phenomenon, we introduce the notion of a *momentum current* or *momentum flux*, which we abbreviate by I_p. It tells us how much momentum is flowing across a system boundary in unit time.

The SI-unit of the momentum current is called Newton (N). As a consequence, the unit of momentum must be N·s (Newton-second).

Consider a block of wood being pushed by a stick on a horizontal surface (Fig.13.5). The block accelerates, it obtains momentum which must flow through the stick to the block. The stick and the block are both under compressional mechanical stress. How strongly the material is stressed is measured by how dense the current is if measured at a surface. The thinner the stick for the same loading, the higher the stress. If the stick is thinner, the same momentum current must flow through a smaller cross section. Mechanical stress is quantified by introducing the *momentum current density* j_p:

$$I_p = A j_p \tag{13.1}$$

The SI-unit of the momentum current density is $[j_p] = \mathrm{N/m^2}$. In other words, the current density measures how the current is distributed over the surface through which it flows. This distribution is a fundamental mechanical quantity which tells us about the state of stress of a material.

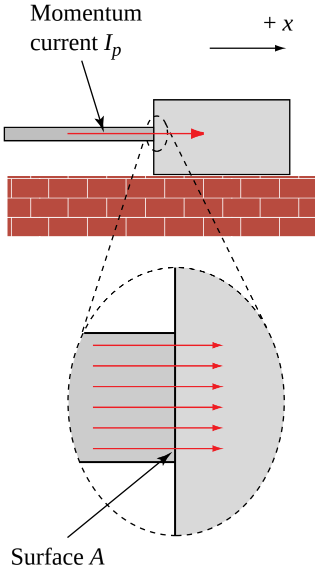


FIGURE 13.5. When bodies interact mechanically by touching, momentum flows through the bodies and across surfaces. The surface density of the momentum current is the measure of mechanical stress.

- *Note that in a material under* compressional stress, *momentum flows in the* positive spatial direction *(Fig.13.5). If the material is under* tension, *momentum flows though it in the* negative *direction*.

Now consider the case of the wooden block again. Assume that we push it in such a way that it slides across the rough surface at constant speed. As far as the stick is concerned, its situation is the same; it is under compression, and it communicates momentum to the block. However, the blocks speed is not supposed to change which means that its momentum cannot change. Therefore, momentum must flow out of the body at just the same rate at which it receives it. Naturally, this is the result of friction at the bottom of the block. Note that the momentum supplied flows sideways out of the body, leading to what is called shearing stress (Fig.13.6).

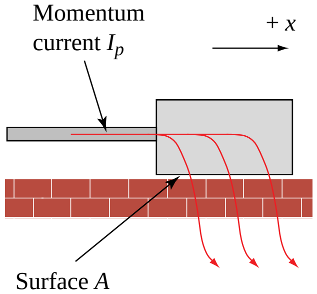


FIGURE 13.6. Shearing stress is the result of the sideways flow of momentum.

- *If momentum flows sideways through a body, this leads to* shearing stress. *Alternatively, this is called tangential stress.*

Pressure as normal stress. We have known a special case of mechanical stress for quite a while already—namely the *pressure of fluids*. If we select a portion of a fluid as a body, we see that this body presses upon the surrounding material. This material may be more fluid, or a part of a container wall. Either way, the situation is just as in Fig.13.5 where the stick pushes the block; we could have used a fluid under pressure to have the same effect upon the block.

There is one notable difference, however. Whereas a solid stick can be used to also set up tangential stress on the surface of another body, a fluid cannot do so, unless it is viscous. Even a viscous fluid cannot transmit momentum obliquely to a surface if it is at rest. Either way, the normal component of the momentum flux across a fluid surface is called the pressure of the fluid. The unit of pressure is the same as that of a momentum current density. Therefore, $\mathrm{Pa} = \mathrm{N/m^2}$.

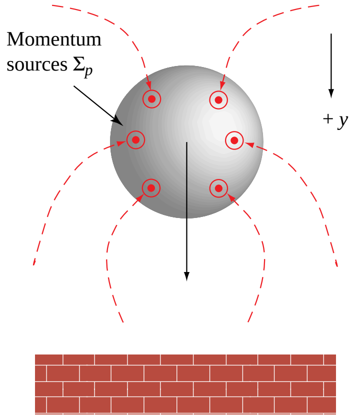


FIGURE 13.7. Momentum flows directly into every part of a body through the Earth's gravitational field. Instead of currents, we have source rates of momentum. The dashed lines are supposed to represent momentum flow through the gravitational field. They are not drawn realistically.

Bodies Falling at the Surface of the Earth

Bodies can be set in motion without direct touch by another object. As an example take a stone falling at the surface of the Earth. Since the body is speeding up, its momentum must be increasing. We say that it receives momentum from the Earth (which therefore moves in the opposite direction toward the stone) *through* the Earth's *gravitational field* (Fig.13.7).

This transfer has completely different properties than conductive transfer. Bodies do not touch; momentum does not flow through bodies, it flows *directly into (or out of) each part* of a body, without influencing other parts. This is a volumetric process, as opposed to a surface process as in conduction. It has very much in common with *radiative transfer* of entropy into and out of bodies (Chapter 10).

As in the case of radiation of heat, we introduce source rates to quantify the transfer of momentum. A *momentum source rate* Σ_p tells us, how much momentum a body receives overall per unit time. It has the same units as that of a momentum flux, namely Newton.

Note that the transfer of momentum directly into and out of bodies does not lead to mechanical stress by itself. There must be a reason for momentum to flow through the material of a body for stress to occur (Fig.13.8).

What we describe here is the result of the interaction of bodies and fields. There are other examples of such processes. Electrically charged bodies react to electric fields, and moving charged bodies or magnetized materials receive momentum through the magnetic field (Chapter 16, and Part V). Just as a body has to be electrically charged to react to an electric field, it has to be "gravitationally charged" to be influenced by a gravitational field. The gravitational charge is the gravitational mass, and the interaction of a stone and the Earth's gravitational field leads to the weight of the body.

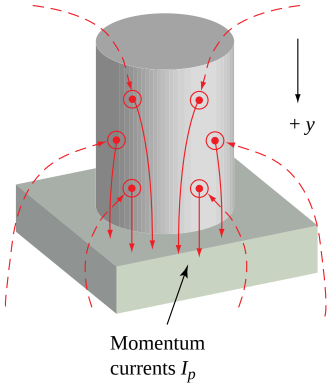


FIGURE 13.8. In a body at rest at the Earth's surface, the momentum supplied through the gravitational field (sources in the body) flows out again through the bottom. The currents flow through the material (conduction), and the body is under compressional stress.

Momentum Transfer with Fluids: Convective Momentum Currents

Momentum can be transferred into and out of systems with flowing fluids. The flowing—and therefore moving—fluid contains momentum. If it flows into or out of a system, it brings or takes some momentum, leading to a momentum current with respect to the system.

Conductive and radiative transfer of momentum keep the material integrity of a body; they do not change the mass of a body. Transfer with fluids—which is called convection—does not. Here, we only study the motion of bodies, not of open systems which accept the flow of fluids. Convective transfer is treated in Part IV of the book.

Q

1. A body is pushed by a beam through which a momentum current of 120 N flows for 10 s. How much momentum is transferred to the body through the beam? If at the same time an amount of momentum equal to 700 N·s flows out of the body, by how much should its momentum have changed?
2. The beam in Question 1 has a cross section of 0.0050 m^2. How large is the mechanical stress in the beam? In which direction of space is momentum flowing?
3. If in Fig.13.7 the positive direction of space is chosen to point upwards, where does momentum come from or go to?
4. Change the positive direction of space in Fig.13.8 to the upward direction. Does the conductive momentum flow through the cylinder change direction? Does this change the stress?

A

1. With a current of 120 N flowing constantly over 10 s, an amount of momentum equal to 120 N · 10 s = 1200 N·s will have been communicated to the body. Since momentum is conserved, the change of the momentum of the system must be Δp = 1200 N·s – 700 N·s = 500 N·s.
2. If the momentum current is uniform over the cross section of the beam, the momentum current density (which is the stress) is $j_p = I_p/A$ = 120 N / 0.0050 m^2 = 24·10^3 N/m^2. The beam is compressed, therefore momentum is flowing in the positive direction of space.
3. Now, the body is falling in the negative direction. Its velocity is becoming more negative in the course of time. Therefore, momentum must leave the body. There will be sinks of momentum rather than sources with respect to the body, and momentum will flow through the gravitational field to the Earth.
4. Since we no have sinks of momentum with respect to the body, momentum must be supplied to it through its bottom cross section resting on the floor. Momentum is transported conductively through the cylinder in the upward (positive) direction. Naturally, the stress is still compressional.

13.3 Momentum, Velocity, and Momentum Capacitance

The momentum of bodies depends upon two factors. First, for a given body, the higher the velocity, the higher the momentum. Second, different bodies moving together at the same speed usually contain different amounts of momentum. There is a quantity which measures the “size” of a body such that the bigger the “size” the more momentum it contains at a given velocity. Note that the “size” cannot be the volume of a body.

Dependence of Momentum Upon the Velocity

Simple measurements of completely inelastic collisions of identical gliders on an air track demonstrate that the momentum of a body must be proportional to its velocity (Fig.13.9):

$$p \sim v \qquad (13.2)$$

Actually, this relation only holds for velocities which are small compared to the speed of light (Section 13.7). Note that since the velocity can either be positive or negative, we expect the same to be true for momentum as well.

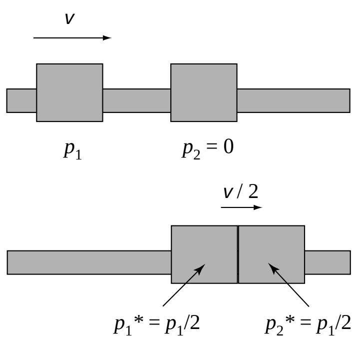


FIGURE 13.9. Two identical gliders collide on an air-track. The first was moving, the second was at rest. After the inelastic collision, the momentum is split among the bodies. Measurements show that the speed of the combined system is half that of the first glider before the collision.

The Momentum Capacitance

Obviously, the factor of proportionality relating momentum and velocity in Eq.(13.2) measures how much momentum a body contains per unit velocity; such a factor has all the properties of a capacitance (Table 13.2):

$$momentum\ capacitance = \frac{p}{v} \qquad (13.3)$$

The momentum capacitance measures how much momentum has to be added to a body so it will have a certain velocity. Since it is harder to set a body with a large momentum capacitance into motion, this factor is also said to measure the *inertia* of the body. For this reason, the momentum capacitance is commonly called the *inertial mass* m_i of the body. In other words, $m_i = p/v$, or

$$p = m_i v \tag{13.4}$$

The inertial mass is given the same unit as the gravitational mass, i.e., $[m_i]$ = kg. The reason for this will be discussed further below. Momentum capacitances of bodies can be inferred from the measurement of the outcome of collisions, or, as we shall see shortly, from the measurement of their gravitational masses.

TABLE 13.2. Capacitances of physical systems

Field	Capacitance
Hydraulics	$K_V = dV/dP$
Electricity	$C = q/U$
Rotation	$J = L/\omega$
Heat	$K = dS/dT$
Motion (Translation)	$m_i = p/v$

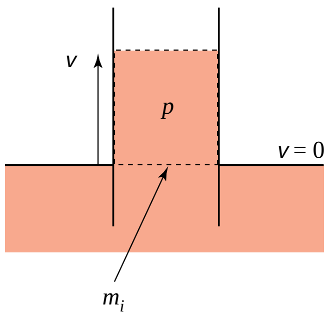


FIGURE 13.10. Fluid image of momentum. The momentum of a body is like the amount of fluid in a container. The level of the fluid and the cross section of the container represent the velocity and the inertial mass of the body, respectively.

The Fluid Image of Momentum

The relation between momentum, velocity, and inertial mass of a body can be cast in the form of a graphical image already used in previous chapters (Fig.13.10). A body is represented by a (straight walled) container whose cross section is taken to be the momentum capacitance, i.e., the inertial mass. The fluid contained in the imaginary tank represents the momentum stored, and the level of the fluid denotes the velocity of the body.

This image is analogous to what we already used for rotational processes (Chapter 5), and very similar to the hydraulic image of other quantities. The drawing in Fig.13.10 shows the velocity of the body with respect to a value of zero. For the purpose of discussing motion here at the surface of our planet, we can associate this special level with the velocity of the Earth. The Earth itself is like a giant reservoir of momentum which supplies or receives quantities of motion without being affected by the processes of ordinary bodies. The velocity of a body, and therefore its momentum, can be positive or negative with respect to the level of this reservoir.

Fluid Image and Momentum Balance in Collisions

The fluid image can be used to visualize collisions between two bodies in a single direction of space (Fig.13.11). The momentum of both bodies is measured with respect to the same observer (which may be taken to be at rest relative to the Earth). It is commonly assumed that—at least right during the collision—the system of two bodies is isolated from the rest of the world, leaving its total momentum untouched. With $p_1 + p_2$ = constant, we have

$$\Delta p_1 = -\Delta p_2 \tag{13.5}$$

This is represented graphically in the fluid or hydraulic image of the collision, and can even be used to solve the problem graphically.

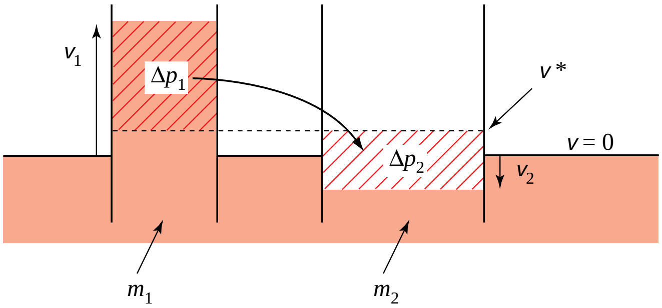


FIGURE 13.11. Completely inelastic collision between two bodies in the fluid image. The second body initially moves in the negative direction. The dashed horizontal line at $v = v*$ denotes the common final velocity of the combined body.

Inertial and Gravitational Mass

Bodies with larger inertia also are heavier. Indeed, this is a strict proportionality and represents one of the most remarkable laws of nature:

$$m_i \sim m_g \tag{13.6}$$

It is commonly called the *law of equivalence of gravitational and inertial mass.* It has been verified to high precision in many different experiments, and the finding has been used as the basis of the general theory of relativity. Even though the phenomena of gravity and inertia are quite different superficially, their measures are given the same values and the same unit, i.e. the kilogram. Naturally, as a consequence of this proportionality, we can use scales to measure the inertial mass of bodies.

EXAMPLE 13.1. Measuring the inertial mass of bodies in collisions.

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a) A glider having unit inertial mass and a velocity of 0.90 m/s collides with a stationary glider. Both move on an air-track. The bodies stick together after the collision, and their velocity is measured to be 0.65 m/s. Determine the inertial mass of the second glider. b) Two astronauts float in a spacecraft. One of them pushes the other whereupon they float apart, the first with a velocity of 0.50 m/s, the second with a speed of 0.60 m/s in the opposite direction. The first is known to have a mass of 80 kg. How large is the mass of the second astronaut?

SOLUTION: a) We can use Fig.13.11, where $v_2 = 0$, to solve the problem. The cross section of the second container is unknown, however, we know the final common velocity of the combined system. Using Eq.(13.5), we can write

$$m_1(v*-v_1) = -m_2(v*-0)$$

or

$$m_2 = -\frac{v*-v_1}{v*}m_1 = -\frac{0.65-0.90}{0.65}1.0\,\text{kg} = 0.385\,\text{kg}$$

b) The process is the reverse of a completely inelastic collision (the astronauts are moving together at first, and as separate bodies after the event). Here, before the process, both bodies are at rest with respect to the observer, meaning that the containers in Fig.13.11 are empty. The event leads to one container being filled at the cost of the other. In other words, momentum is pumped from one to the other container. Again we use Eq.(13.5):

$$m_1(v_1*-0) = -m_2(v_2*-0)$$

which leads to $m_2 = -v_1*/v_2*m_1 = -0.50/(-0.60)\cdot 80$ kg = 66.7 kg.

■

13.4 Momentum and Energy in Collisions

The fluid image of momentum can be used to explain part of the role of energy in the motion of bodies. This is particularly useful for calculating the outcome of collisions, since the law of balance of momentum alone does not always suffice to obtain results.

Energy of a Moving Body

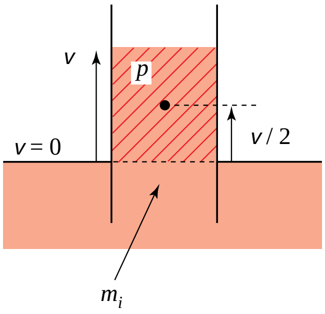


FIGURE 13.12. The energy of a body associated with its motion is like the energy stored with a fluid in a tank.

As with all other phenomena, energy accompanies also the processes of translational motion. In particular, a moving body stores some energy, just as a hot or charged body does. Again, the hydraulic image lets us visualize the energy associated with motion. Since we view the "content" of a container as the momentum of the body, and since this content has to be filled into the container—a process for which we need energy—the energy is equal to the product of the level of the center of the content (which is v/2) and the content (Fig.13.12):

$$W = \frac{1}{2} v\, p = \frac{1}{2} m_i v^2 \tag{13.7}$$

The energy stored by a body as a result of its motion is often called *kinetic energy.* Note that the name does not mean that this is a special type of energy. Rather, it tells us something about the system storing the energy.

Releasing and Binding Energy in Collisions

In collisions, momentum flows from bodies of high velocity to those of lower velocity, or vice-versa. Therefore, energy must be released or bound in these processes.

The fluid image shows also how much energy has been released or bound from the start to a given moment during the collision. The amount of energy released corresponds to the product of the amount of momentum transferred up to that point (p_e) and the average "fall" of level (Δv) of the transferred momentum (Fig.13.13):

$$W_{released} = -\Delta v \left| p_e \right| \tag{13.8}$$

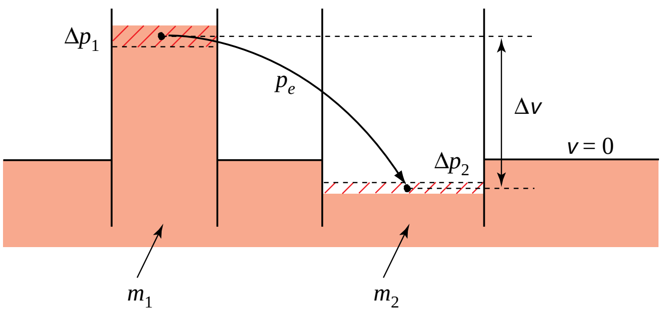


FIGURE 13.13. Energy released in the "fall" of momentum is shown in the fluid image of a collision. The figure shows the first short time span during a collision.

Energy released can also be bound. In an elastic collision, for example, momentum is pumped to higher levels after the bodies have reached a common velocity during the first phase of the collision. If all the energy released (up to the point where the speeds of the bodies are equal) is bound for further motion, the collision is said to be completely elastic. Naturally, energy which is not bound in such a process is dissipated, i.e., entropy is produced. In a completely inelastic collision—which results in a single body—all the energy released is dissipated.

Collisions in a Single Direction of Space

Combining the laws of balance of momentum and energy lets us calculate the outcome of general processes of the collision of two bodies in a single spatial dimension. Consider a completely elastic collision. Momentum falls—and energy is released—until the common speed of a complete inelastic collision is reached. (See the inelastic line in Fig.13.14). Since the collision is not dissipative, the energy released is bound in

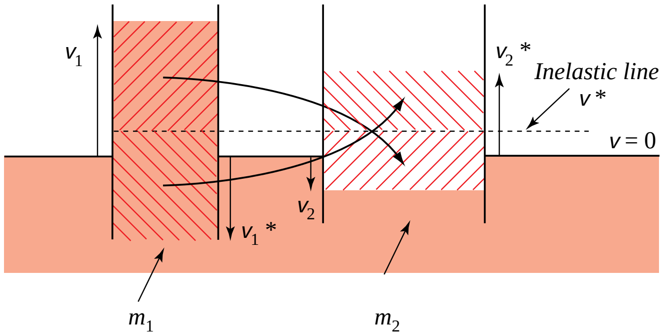


FIGURE 13.14. Fluid image of the completely elastic collision between two bodies. The second one initially moved in the negative direction. The energy released during the first phase of the collision (up to the inelastic line) is used to pump again as much momentum from the first to the second body.

pumping the same amount of momentum "uphill" during the second stage of the collision. The final state is equal to the initial one mirrored at the inelastic line.

Dynamic Model of Momentum and Energy Balances

In previous chapters, we have used the system dynamics representation of processes to learn more about them. If we write the law of balance of momentum in terms of combinations of stocks and flows of momentum, and then manage to express the momentum flows, we can compute the processes of motion. In this chapter, however, we will only be able to solve part of the problem, since we still lack knowledge of concrete relations for momentum currents in collisions.

As an example of how to introduce system dynamics modeling to deal with collisions, again consider the completely inelastic interaction between two bodies moving in a straight line (Fig.13.15). You will see that—by simply assuming a form of the momentum flux during the collision—we will be able to determine the final velocity and the amount of energy released.

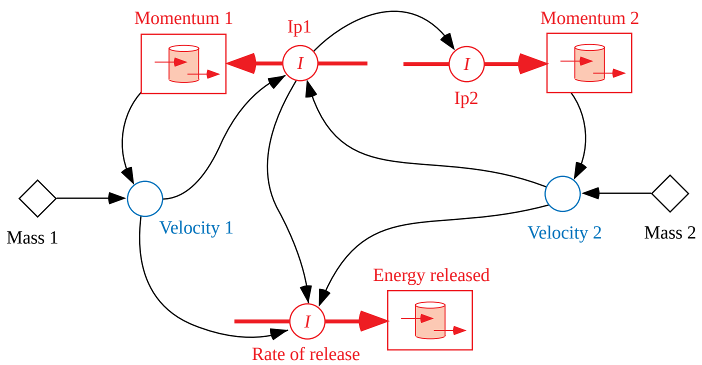


FIGURE 13.15. System dynamics model of a completely inelastic collision of two bodies moving in a straight line. Momentum balances are expressed by stocks and flows, velocities are calculated like water levels, and the energy released by the fall of momentum is obtained by integrating the rate at which it is released.

Since there are two bodies, we write the law of balance of momentum twice. There is only one current, however, from the first to the second body, just as in communicating water tanks. We introduce two momentum fluxes, one for each body, where the fluxes are equal in magnitude but have the opposite signs. The velocities are calculated just like water levels in hydraulic models (Chapter 1).

Lacking a proper law for the momentum current, we give it a constant average value such that the momentum is exchanged in a short period equaling the duration of the collision. We simply set the current equal to zero as soon as the velocities have become equal. This ensures that the velocities of the bodies stop changing when the collision is over. Naturally, by doing this we still do not have a realistic model of how the collision proceeds in time.

While momentum flows from the first to the second body, energy is released at a rate which is the in instantaneous form of Eq.(13.8), i.e.,

$$P = -\Delta v \left| I_p \right| \quad \textbf{(13.9)}$$

Imagine the energy released to be stored in an intermediate "storage device" to be used later in a follow up process. Therefore, the release of energy—and its later use—can be represented by a kind of balance of energy (Fig.13.15).

EXAMPLE 13.2. A partly inelastic collision.

A small truck bumps into a car from behind. The truck and the car were moving at 60 m.p.h. and 40 m.p.h. before the collision, and their masses are 1500 kg and 800 kg, respectively. If half of the energy released is dissipated, what should the final velocities of the vehicles be after the collision?

SOLUTION: First, we calculate the inelastic line (the velocity attained in a completely inelastic collision). From Eq.(13.5) we find:

$$m_1\left(v^* - v_1\right) = -m_2\left(v^* - v_2\right)$$

or

$$v^* = \frac{m_1 v_1 + m_2 v_2}{m_1 + m_2} = \frac{1500 \cdot 26.7 + 800 \cdot 17.8}{1500 + 800}\frac{\text{m}}{\text{s}} = 23.6\frac{\text{m}}{\text{s}}$$

Now we can determine the amount of energy released. The momentum which is exchanged up to this point is

$$\left|p_e\right| = \left|m_1\left(v^* - v_1\right)\right| = \left|1500(23.6 - 26.7)\right|\text{N}\cdot\text{s} = 4.65\cdot 10^3\,\text{N}\cdot\text{s}$$

The average velocities of the vehicles during this first phase of the collision are 0.5(26.7+23.6) m/s = 25.15 m/s, and 0.5(17.8+23.6) m/s = 20.70 m/s, respectively. With Δv = 20.70 m/s – 25.15 m/s = – 4.45 m/s, the amount of energy released is

$$W_{released} = -\Delta v\left|p_e\right| = -(-4.45)\cdot 4.65\cdot 10^3\,\text{J} = 20.7\cdot 10^3\,\text{J}$$

Half of this is used to pump an additional amount of momentum from the truck to the car. This leads to two relations

$$m_1\left(v_1^* - v^*\right) = -m_2\left(v_2^* - v^*\right)$$

$$W_{released}/2 = \left[0.5\left(v_2^* + v^*\right) - 0.5\left(v_1^* + v^*\right)\right]m_2\left(v_2^* - v^*\right)$$

whose solution is v_1^* = 21.4 m/s and v_2^* = 27.7 m/s. The relations can be represented graphically in Fig.13.14. If the collision had been completely elastic, the car would have had a final velocity of 29.4 m/s, whereas the truck would decelerate to a speed of 20.5 m/s.

■

13.5 The Momentum of Extended Bodies: The Center of Mass

The momentum of a body can be expressed in terms of its velocity and its mass. With extended bodies there is a problem defining its velocity. In general, each point of a body has a different velocity. Consider a sphere rolling on a flat surface (Fig.13.16), or a spring being stretched, and the problem becomes obvious. Therefore, we must find a point which represents the translational motion of an extended body.

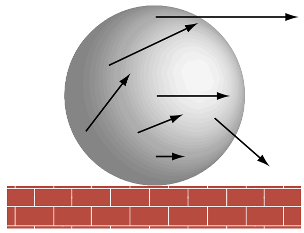


FIGURE 13.16. Different points of a rolling sphere have different velocities. Still, there is a special point of the body which we can take to represent the translational motion.

Velocity of the Center of Mass

The problem is solved as follows. We will define the representative velocity of a body such that the momentum calculated on its basis is equal to the sum of all the quantities of motion of its parts. As the simplest possible example of a composite body, consider a system consisting of just two very small bodies having masses m_1 and m_2, respectively, moving in a straight line (Fig.13.17). The total momentum of the system is

$$p = m_1 v_1 + m_2 v_2 \tag{13.10}$$

On the other hand, if we look at the system as a single body of mass $m_1 + m_2$ moving at a certain velocity v_{CM}, its momentum would be

$$p = (m_1 + m_2) v_{CM} \tag{13.11}$$

The velocity representing the translational motion of the system is called the velocity of its center of mass. From Eqs.(13.10) and (13.11) we have

$$v_{CM} = \frac{m_1 v_1 + m_2 v_2}{m_1 + m_2} \tag{13.12}$$

This definition of the velocity of the center of mass can be extended to any number of particles in one or more spatial dimensions, as well as to spatially continuous distributions of mass.

Consider again the two mass points used above (Fig.13.17). We can define a special point of the system by

$$x_{CM} = \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2} \tag{13.13}$$

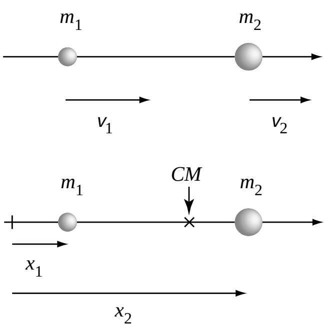


FIGURE 13.17. A system consisting of two particles. The total momentum defines the velocity of the center of mass, and the particles positions and masses define the position of the center of mass.

which is called the *center of mass* of the system. Obviously, the center of mass moves at the speed v_{CM}. To see this, we only have to calculate the velocity associated with the center of mass from Eq.(13.13).

Properties of the Center of Mass

The center of mass of an extended system has some important properties which will be listed here. Actually, all the claims made can be proved.

- The motion of a body—and therefore its momentum—is always measured with respect to an observer. If we take an observer moving with the same velocity as that of the center of mass given in Eq.(13.12), the observer measures a momentum of zero for the system (Fig.13.18).

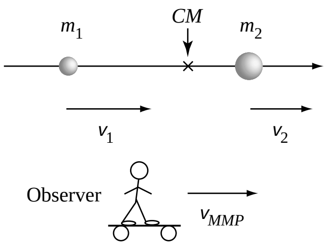


FIGURE 13.18. An observer moving with the center of mass measures a momentum of zero for the system.

- If a system is mechanically isolated from the environment, i.e., if there are no momentum transports from the system to the environment or vice-versa, its momentum cannot change. Therefore, the velocity of the center of mass must remain constant.
- The center of mass of a system does not have to coincide with a material part of the system.
- The velocity of the center of mass of a system composed of particles corresponds to the velocity the system would attain after all of its parts merged into a single body as a result of multiple completely inelastic collisions.

Observers and the Momentum of Bodies

Motion is always determined with respect to a chosen observer. Velocities are measured with respect to such an observer. Therefore, the momentum of a body also depends upon the observer which has been chosen for a particular case.

We said that the velocity of the center of mass of an isolated system must remain constant. This is not true for an observer moving at increasing or decreasing speed. Therefore, unless otherwise noted, we will only choose observers for whom the claim is true. These are called *inertial observers* or *inertial frames of reference*. Looking at motion from non-inertial frames requires an extension of the rules for finding momentum currents which will be discussed briefly in Chapter 16.

Q

1. Why is the motion of the center of mass representative of the motion of an extended system?
2. Why does the velocity of the center of mass of a mechanically isolated system remain constant? What kind of motion is therefore possible for the center of mass of such a body?

A

1. The velocity of this point multiplied by the total mass of the system is equal to the total momentum of the body.
2. Mechanical isolation means that there are not momentum transports with into or out of the body. Since momentum cannot be created or destroyed, the quantity of motion of the system must remain constant. The motion of such a system is either in a straight line with constant speed, or the body is at rest with respect to the observer.

EXAMPLE 13.3. Center of mass of the Earth-Moon system.

Where is the center of mass of the system composed of Earth and Moon? The mean distance between Earth and Moon is 384,000 km, the masses of the two bodies are $6.0 \cdot 10^{24}$ kg and $7.4 \cdot 10^{22}$ kg, respectively. Assume an observer for whom this center of mass moves at a speed of 30 km/s. What is the momentum of the system?

SOLUTION: The center of mass of the system is found on the straight line connecting the centers of Earth and Moon (Fig.13.17). Introduce a coordinate system (x-coordinate) along this line having its origin at the center of the Earth. Therefore, we have $x_1 = 0$ m and $x_2 = 3.84 \cdot 10^8$ m. Using Eq.(13.13), we find that

$$x_{CM} = \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2} = \frac{6.0 \cdot 10^{24} \cdot 0 + 7.4 \cdot 10^{22} \cdot 3.84 \cdot 10^{8}}{6.0 \cdot 10^{24} + 7.4 \cdot 10^{22}} \text{m} = 4680\,\text{km}$$

Since the radius of the Earth is approximately 6400 km, the center of mass of the Earth-Moon system lies inside the Earth.

■

13.6 Collisions in Two and Three Dimensions

Collisions in two—or three—dimensions demonstrate a centrally important feature of momentum: there are two—or three—*independent components of momentum*, i.e., components for which the rules found so far hold independently. Alternatively, we may say that momentum is a *vector* quantity.

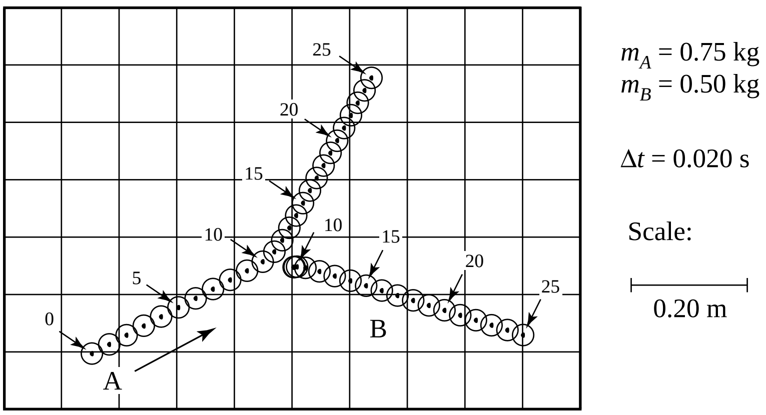
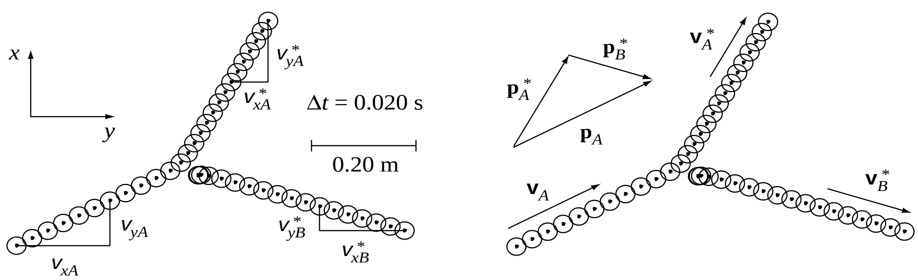


FIGURE 13.19. Elastic collision of two magnetized pucks on an air-table. Puck A moves from the left and collides—without touching—with Puck B which was at rest initially. This is a drawing made according to a stroboscopic photograph of the collision (single frames at a time interval of 0.020 s).

Component and Vector Representations

Let us investigate the collision shown in Fig.13.19. First, we determine the velocities of the bodies before and after the collision in the *x*- and *y*-directions (Fig.13.20a). This is done by measuring the distance travelled in either direction during a certain number of successive time intervals (here there are six with a total interval of 0.12 s). These components of distances are divided by the total time interval which yields the components of the velocities. Finally, the velocity components are multiplied by the proper masses of the bodies which leads to the components of momentum (Table 13.3).



a. Component representation *b. Vectorial representation*

FIGURE 13.20. Component and vectorial representations of a collision in two dimensions. Table 13.3 show that each component of momentum in the x-y coordinate system is conserved individually. The triangle formed of the vectors of momentum demonstrates that the sum of the momentum vector(s) before the collision is equal to their sum after the event.

The components of momentum are calculated from the components of velocity by multiplying the latter by the proper mass, i.e.,

$$p_{xA} = m_A v_{xA}$$
$$p_{xB} = m_B v_{xB} \qquad (13.14)$$

for the x-components before the collision (Fig.13.21).

TABLE 13.3. Components of momentum in the collision of Fig.13.19

		Δ(position) m	Velocity m/s	Momentum N·s
A before	x-direction	0.178	1.49	1.11
	y-direction	0.095	0.79	0.59
B before	x-direction	0.000	0.00	0.00
	y-direction	0.000	0.00	0.00
A after	x-direction	0.070	0.59	0.44
	y-direction	0.129	1.07	0.80
B after	x-direction	0.163	1.36	0.68
	y-direction	– 0.051	– 0.42	– 0.21

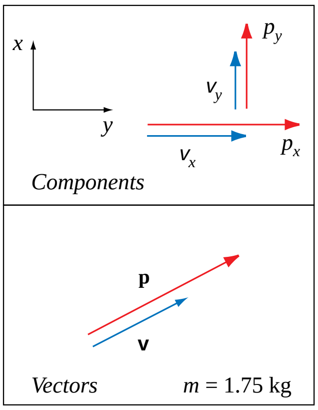


FIGURE 13.21. Multiplication of velocity components or vectors by the mass of the body. The velocity components or vectors simply are stretched by the factor represented by the mass.

Having two or three independent components of velocity and momentum means that these quantities are vectors. Fig.13.20b demonstrates that the vector of momentum is a conserved quantity. We show this by determining the velocity vectors of the bodies before and after the collision by drawing an arrow in the direction of motion whose length represents the magnitude of the velocity—called the speed—of a body. The velocity vectors are then multiplied by the value of the mass of the associated body which yields the momentum vector:

$$\mathbf{p} = m\mathbf{v} \qquad (13.15)$$

This relation is demonstrated in Fig.13.21.

Momentum and Energy in Collisions

Inspection of the values in Table 13.3 proves that—within the bounds of errors made by measuring the quantities—the sum of the x-components of momentum of the bodies before the collision is equal to their sum after the process; the same is true for the y-component:

$$p_{xA} + p_{xB} = \overset{*}{p}_{xA} + \overset{*}{p}_{xB}$$
$$p_{yA} + p_{yB} = \overset{*}{p}_{yA} + \overset{*}{p}_{yB} \qquad (13.16)$$

■ *In two (three) spatial dimensions, there exist two (three) independent components of momentum. For each component, the properties known for a single one—storage, transfer, and conservation—hold independently.*

Again, within measuring errors, we see that the momentum of the bodies is conserved in the collision (Fig.13.20):

$$\mathbf{p}_A + \mathbf{p}_B = \mathbf{p}_A^* + \mathbf{p}_B^* \tag{13.17}$$

In analogy to the conclusion for the components reached above, we can say that momentum, as a vector, satisfies the rules known for a single component:

■ *In two (three) spatial dimensions, momentum behaves like a vector. Momentum is stored, i.e., the stored quantity is a vector, momentum flows, i.e., momentum fluxes are vectors, and momentum is conserved, i.e., the law of balance of momentum is a vector equation.*

Energy in collisions. Energy is not a vectorial quantity, it is a *scalar*. Therefore, there are no *x*- or *y*-components of energy, and the energy of a body has to be calculated from its speed, i.e., from the absolute value

$$|\mathbf{v}| = \sqrt{v_x^2 + v_y^2} \tag{13.18}$$

The energy of the moving body is

$$W = \frac{1}{2} m |\mathbf{v}|^2 \tag{13.19}$$

in analogy to Eq.(13.7). The values obtained from the observation of the collision shown in Fig.13.19 and listed in Table 13.3 prove that the energy of the moving bodies is conserved as well since the collision is completely elastic; there is no dissipation.

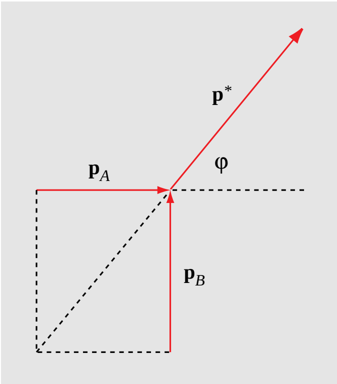
Q

1. If in Fig.13.19 the perpendicular lines denoting a planar frame of reference are all rotated by the same angle, what happens to the components of momentum (or velocity) and the momentum vectors of the bodies involved in the collision?
2. If body B moved in the same direction after the collision as did body A before (Fig.13.19), in which direction would body A have to move after the event?

A

1. Rotating the frame of reference does not change the appearance of the collision, i.e., the tracks of the bodies and the distances between successive points, at all; neither is the time interval changed. The way vectors are determined shows that these are not changed by the rotation. The components, however, change their values. Still, even though the individual values are different, the components of momentum still satisfy the rules observed in this chapter. The laws of nature do not change.
2. The vector sum of momentum before the collision must point in the same direction as that after the process. Since the momentum of B already points in this direction, A may not have a different direction. Body A must therefore move in the same line as it did before the collision, either forward or backward.

EXAMPLE 13.4. Inelastic collision in 2D.



Two cars collide at an icy intersection at right angles. The first car, having a mass of 700 kg, moved at a speed of 50 km/h, while the second one, having a mass of 800 kg, moved at a speed of 65 km/h. In the collision, the cars become entangled. Determine the direction and the speed of the cars after the collision. How much energy has been dissipated?

SOLUTION: We use a vectorial representation of the collision (components would work just as well). The momenta of the cars have absolute magnitudes of

$$|\mathbf{p}_A| = m_A|\mathbf{v}_A| = 700 \cdot 50/3.6\,\mathrm{N \cdot s} = 9722\,\mathrm{N \cdot s}$$
$$|\mathbf{p}_B| = m_B|\mathbf{v}_B| = 800 \cdot 65/3.6\,\mathrm{N \cdot s} = 14440\,\mathrm{N \cdot s}$$

The directions are shown in the accompanying graph. Since momentum is conserved in the collision, we have

$$\mathbf{p}_A + \mathbf{p}_B = \mathbf{p}^*$$

The magnitude and the direction of the resulting momentum vector are

$$|\mathbf{p}^*| = \sqrt{|\mathbf{p}_A|^2 + |\mathbf{p}_B|^2} = \sqrt{9722^2 + 14440^2}\,\mathrm{N \cdot s} = 17410\,\mathrm{N \cdot s}$$

and

$$\varphi = \arctan\left(|\mathbf{p}_B| / |\mathbf{p}_A|\right) = \arctan(14440/9722) = 56.1°$$

respectively. The speed of the cars moving after the collision is determined by

$$|\mathbf{v}^*| = \frac{1}{m_A + m_B}|\mathbf{p}^*| = \frac{17410}{700+800}\frac{\mathrm{m}}{\mathrm{s}} = 11.6\frac{\mathrm{m}}{\mathrm{s}}$$

The energy dissipated is equal to the difference of the energies of the bodies due to motion before and after the collision:

$$\begin{aligned} W_{diss} &= (W_A + W_B) - W^* \\ &= \left(\frac{1}{2}m_A|\mathbf{v}_A|^2 + \frac{1}{2}m_B|\mathbf{v}_B|^2\right) - \frac{1}{2}(m_A + m_B)|\mathbf{v}^*|^2 \\ &= 0.5\left[700 \cdot 13.89^2 + 800 \cdot 18.06^2 - 1500 \cdot 11.6^2\right] = 97\,\mathrm{kJ} \end{aligned}$$

■

13.7 Motion at Very High Speed

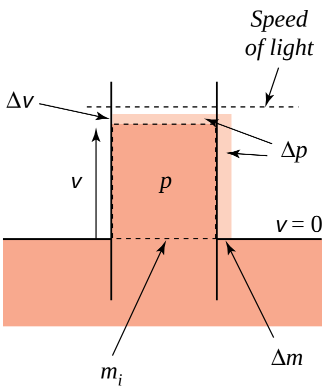


FIGURE 13.22. Increasing the momentum of a body only partially leads to an increase of speed. The mass of the body increases as well, particularly at speeds close to the speed of light. Since the energy is increased as well, we find an important relation between energy and mass.

We may wonder what happens when we give a body more and more momentum. According to what we know so far, its speed increases all the time. In the fluid image of Fig.13.10 the fluid level simply continues to increase.

Motion at very high speeds demonstrates features which are not known at lower speed. In particular, the speed of light cannot be surpassed by a body. Since it is still possible to give more momentum to a body moving at a speed close to that of light, we must conclude that increasing the momentum leads to an increase in the momentum capacitance, i.e., the mass, of the body (Fig.13.22).

Giving a body more momentum also means giving it more energy. Therefore, there must be a close relationship between the energy and the mass of a physical system. This is known as the *equivalence of mass and energy* in the theory of relativity.

Light Carries Momentum

One of the most important observations about light is that it also behaves as if it were a stream of particles—called photons—with properties which in some ways are like those of ordinary bodies. Photons carry momentum, just like bodies, but they cannot move at speeds other than the speed of light. Observers moving at totally different speeds with respect to each other will all measure the same speed for the propagation of light.

Since photons carry momentum, we introduce the momentum capacitance m of light and write

$$p = mc \qquad \textbf{(13.20)}$$

where c stands for the speed of light ($c = 2.998 \cdot 10^8$ m/s). Since photons only exist at this speed, adding more momentum to light means adding energy according to

$$W = c\,p \qquad \textbf{(13.21)}$$

(See Fig.13.23.) If we combine Eqs.(13.20) and (13.21), we must conclude that

$$W = mc^2 \qquad \textbf{(13.22)}$$

In other words, the momentum capacitance and the energy of light are directly related. The factor converting one value into the other is the square of the speed of light.

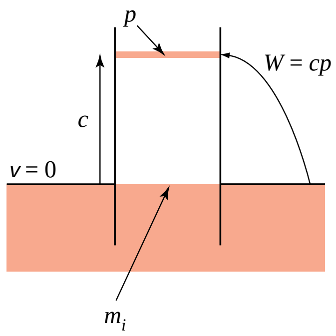


FIGURE 13.23. Creating light having momentum p requires a quantity of energy equal to cp (to "lift" the momentum to the speed c).

Momentum, Mass, and Energy of Ordinary Bodies

Now let us discuss ordinary bodies. It is known that a material body cannot reach or surpass the speed of light. However, as it comes closer to this limit it continually behaves more like a photon. In particular, adding more momentum invariably takes an amount of energy calculated by Eq.(13.21) for photons, as can be seen from either Fig.13.22 or Fig.13.23.

What are the properties of bodies if they start behaving more like photons if moving at very high speeds? Most important is the momentum-speed relation which introduces the momentum capacitance. If we use Eq.(13.22) for the mass and write

$$p = \frac{W}{c^2} v \qquad \textbf{(13.23)}$$

then this relation changes to the one for photons as the body comes close to the speed of light. It is equivalent to Eq.(13.21) for light if we use $v = c$ for the latter.

Let us now consider a process of adding a small portion of momentum Δp to a body moving at speed v. This requires adding an amount of energy $\Delta W = v\Delta p$. Since, according to Eq.(13.22), we also have $\Delta W = c^2\Delta m$, we conclude that

$$\Delta m = \frac{1}{c^2} v \Delta p = \frac{1}{c^2}\frac{p}{m}\Delta p \quad \Rightarrow \quad c^2 m \Delta m = p \Delta p$$

Integration of this equation yields

$$c^2\left(m^2 - m_o^2\right) = p^2 \qquad \textbf{(13.24)}$$

(See Fig.13.24.) Here, m_o denotes the mass of the body when it is at rest. As we have seen in Fig.13.22, the mass of the body increases as momentum is added to it, and our

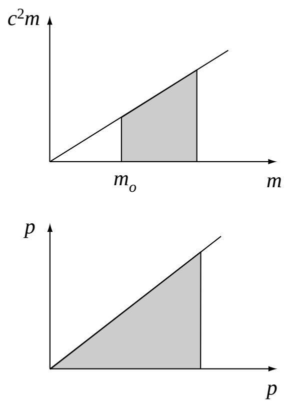


FIGURE 13.24. The quantity c^2m is integrated over m from m_o to m, whereas the quantity p is integrated over p starting at 0.

result tells us how this happens. The relation can be expressed also in terms of the energy of the body:

$$W^2 = c^2 p^2 + \left(m_o c^2\right)^2 \quad (13.25)$$

This is a most remarkable result as we can see by considering its limit for a body at rest. Such a system does not have any momentum which means that

$$W(v = 0) = m_o c^2 \quad (13.26)$$

In other words, a body at rest still possesses energy equal to the product of the *rest mass* m_o and the square of the speed of light. We call it the *rest energy* W_o. This is the source of the interpretation that energy and mass are one and the same physical quantity which is conserved. Adding energy to a body makes it heavier and more inert, which also happens if we increase its mass. Remember that we used this property of energy to argue that the quantity responsible for making a body warm cannot be energy (Chapter 8).

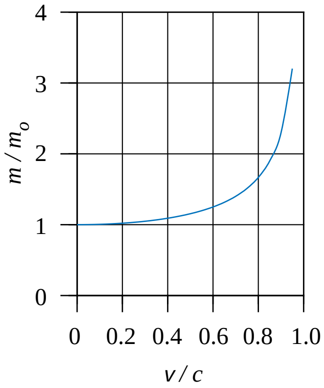


FIGURE 13.25. As the speed of a body increases and comes close to the speed of light, its mass increases ever more sharply.

■ *For ordinary bodies and for light, and for any velocity ranging from zero to the speed of light, the energy and the mass of the system are the same quantity. Their values are related by*

$$W = mc^2 \quad (13.27)$$

We can show now how the mass of a body depends upon its speed. Set $p = mv$ in the relation between momentum and mass, i.e., in Eq.(13.24), and solve for m:

$$m = \frac{m_o}{\sqrt{1 - v^2/c^2}} \quad (13.28)$$

(Fig.13.25). This proves that the mass of an ordinary body would become infinite if it were to move at the speed of light which, therefore, is impossible.

Geometrical Consequences

There are some fascinating geometrical consequences of the properties of motion at high speeds. Since light always has the same velocity viewed from different frames of reference, and since bodies cannot reach or surpass this speed, the rules for adding velocities must be different from those we ordinarily know. At ordinary speeds, two cars heading toward each other will collide at a relative speed which is the sum of the individual values. This can no longer hold for speeds close to the speed of light. For example, if a cosmic ray particle moves toward the Earth at a speed approaching that of light, a different observer moving toward the particle at high speed relative to the Earth will still measure a velocity of the particle which is smaller than the speed of light.

Furthermore, space and time have properties we would not expect from everyday life. In particular, measurements of distances and time intervals depend upon the motion of the observer, with different observers reporting different distances and times. These counter-intuitive properties of space and time have been confirmed again and again in high energy physics laboratories, where elementary particles are made to collide at speeds approaching c. Space and time really are not what they seem to be at first sight. Therefore it is all the more surprising how far we can go with our simplified views of everyday life.

13.8 The Particle Model of the Monatomic Ideal Gas

Physical processes are the result of the storage and transfer of some fundamental quantities such as charge, entropy, or momentum. We have taken this view and combined it with an image of a particular kind—namely that these quantities and the systems they are contained in or flow through are spatially continuous. This is the fundamental assumption of what is known as continuum physics. It has served us well so far in finding out about macroscopic phenomena, and it will continue to guide us through much of our investigations.

There exists a complementary view that physical phenomena are the result of the interplay of countless microscopic processes undergone by the microscopic constituents of matter and radiation. In crude terms, it is sometimes said that processes are the result of the motion of the little particles the world is made out of.

In physics, two theories—*kinetic theory* and *statistical mechanics*—have been developed which aim to shed light on the properties of matter and radiation by considering either the motion of many particles, or average properties of countless numbers of molecules, atoms, photons, ore even smaller constituents of physical systems. Aided by a theory of the behavior of small entities—*quantum physics*—it has increased our knowledge of the world around us tremendously.

Despite their importance and successes, kinetic theory and statistical mechanics do not replace the continuum view of nature when it comes to explaining much of what we are confronted with in science and engineering. They simply are not the right theories to deal with macroscopic phenomena in a practical manner. However, they yield much important information about certain special properties of matter and radiation which is then used in macroscopic models of physical processes. In this sense, kinetic theory and statistical mechanics on the one hand, and continuum physics on the other, are complementary views of how nature operates.

Kinetic theory and statistical mechanics face a severe problem in an introductory physics course: with the exception of a very few applications, they are much too complex. Still, it is important to see in what sense the microscopic view of nature can aid us in understanding also the macroscopic aspects.

The classical physics of collisions of simple particles dealt with in this chapter can be used to present a first look at the utility of kinetic theory. Looking at a monatomic ideal gas such as helium as a collection of individual tiny elastic bodies reveals relations between pressure of the gas and the energy of the particles, or between their average energy (due to motion) and the temperature of the gas. The latter relation also yields a derivation of the temperature coefficient of energy of the monatomic ideal gas which agrees very well with observation.

The Pressure of a Collection of Independent Elastic Particles

In n moles of a gas there are $N = nN_A$ particles, where $N_A = 6 \cdot 10^{23}$ particles/mol is the Avogadro number (Chapter 6). Consider N such particles in a cubic box having sides of length d (Fig.13.26). The particles strike a certain wall at time intervals $\Delta t_i = 2d/v_{xi}$. Here, the index i is used to number the particles, and we consider the x-component of the motion of particles striking the right wall of the container in Fig.13.26. With each collision, a particle transfers an amount of momentum

$$p_{ei} = 2mv_{xi}$$

to the wall. m is the mass of a single particle.

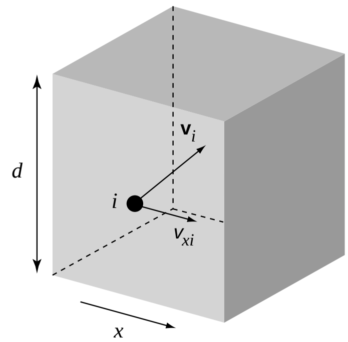


FIGURE 13.26. Particles inside a box collide with the walls. The transfer of momentum to the walls creates the pressure of the gas upon the enclosure.

The pressure of the gas leads to the stress on the walls. In fact, the pressure is equal to this stress (Section 13.2), which is equal to the amount of momentum transferred per time per surface area. For N particles this is

$$P = \sum_{i=1}^{N} \frac{p_{ei}}{d^2 \Delta t_i} = \sum_{i=1}^{N} \frac{2mv_{xi}v_{xi}}{d^2 2d} = \frac{2}{V}\sum_{i=1}^{N}\frac{1}{2}mv_{xi}^2$$

V stands for the volume of the container. Since, for particle i,

$$v_i^2 = v_{xi}^2 + v_{yi}^2 + v_{zi}^2$$

and since the direction into which the particles are moving are independent of each other, we expect that on average

$$v_{xi}^2 = v_{yi}^2 = v_{zi}^2 \quad \Rightarrow \quad v_{xi}^2 = \frac{1}{3}v_i^2$$

Therefore, the pressure of the gas turns out to be

$$P = \frac{2}{3V}\sum_{i=1}^{N}\frac{1}{2}mv_i^2 = \frac{2}{3V}W \tag{13.29}$$

W is the total energy of the gas. This is so since the particles only have energy due to their translational motion (they do not have any other characteristics that could give them more energy on other grounds).

Energy and Temperature

If we combine the result of the particle model with the equation of state of the ideal gas, $PV = nRT$ (Chapter 12), we obtain

$$\frac{2}{3V}WV = nRT \quad \Rightarrow \quad W = \frac{3}{2}NkT$$

Here we have introduced the Boltzmann constant

$$k = R/N_A = 1.3807 \cdot 10^{-23}\ \text{J/K} \tag{13.30}$$

which is also called the gas constant for a single particle. Therefore, the average (kinetic) energy of a single particle is $3/2kT$, which means that

$$\frac{1}{2}m\langle v^2 \rangle = \frac{3}{2}kT \tag{13.31}$$

Therefore, for the monatomic gas, the average of the square of the particle speed is directly proportional to the temperature of the gas.

We can not derive the value of the *temperature coefficient of energy* of the monatomic gas. According to definition, $C_V = \partial W / \partial T$. For our gas we obtain

$$C_V = \frac{\partial}{\partial T}\left(\frac{3}{2}nRT\right) = \frac{3}{2}nR \tag{13.32}$$

This shows that the molar temperature coefficient of energy is $3/2R$ (Chapter 12).

Questions

1. Considering that a body can move in a complicated manner—combining rotation and translation—how would you define translational motion? What characterized translation?
2. How could you detect if a certain body violated the law of equivalence of inertial and gravitational mass?
3. Two coins having the same mass and lying on top of each other are in free fall. Should there be momentum flowing across the surface where they touch? What if the upper coin had a gravitational mass twice the inertial mass? What if the gravitational mass of the lower coin were twice the value of its inertial mass?
4. Does the velocity of the center of mass of two bodies change after a collision? What does it take to change the motion of the center of mass of a system?
5. A ball strikes a wall perpendicularly, and bounces off after an elastic collision. It had a certain momentum before the collision. How much momentum is transferred to the wall? What does the wall do with the momentum? What happens to the energy of the ball?
6. During a one-dimensional collision, two cars lose momentum because of friction. How does Eq.(13.5) change?
7. How does a rocket work?
8. In an explosion, pieces of a body fly off in different directions. How—and where from—do they get their momentum?
9. Why should the mass of a body increase as its speed approaches the speed of light?
10. The pressure of a gas exerted upon the walls of a container can be viewed as the result of momentum transfer of particles to the wall. We know that the pressure is a quantity which exists throughout the body of the gas. Can you use the kinetic theory to explain the pressure in the interior of the gas where there are no walls?

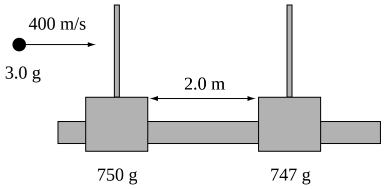
Exercises

1. Estimate the momentum of the following bodies. a) The Earth on its orbit around the sun. b) A car with four passengers moving at 60 m.p.h. c) The bullet of a hand gun. d) A mosquito in full flight.
2. A glider having a mass of 2.0 kg is pulled from rest across an air-track with a string for 3.0 s. The momentum current through the string measures 0.50 N. a) Determine the tensile stress in the string. The diameter of the string is 1.0 mm. b) How much momentum has been flowing into the glider? c) Determine the change of momentum of the glider. d) What is the final velocity of the body?
3. The stress in a rope having a diameter of 2.0 cm is $4.0 \cdot 10^5$ N/m^2. Determine the momentum current flowing through the rope.
4. A heavy block rests on top of a table. Momentum flows at a rate of 200 N from the gravitational field into the block, and at the same rate from the field into the table. a) Determine the momentum current from the block to the table. b) How large is the total momentum current through the four legs of the table. c) Each leg has a circular cross section with a radius of 1.5 cm. What is the stress in each leg?
5. Electrons are accelerated in a CRT through a voltage of 2.0 kV. What is the energy, the speed, and the momentum of a single electron? The mass of an electron is $9.11 \cdot 10^{-31}$ kg. Neglect relativistic effects.
6. A ball strikes a wall perpendicularly, and bounces off after an elastic collision. Its mass and speed before the collision are 0.40 kg and 10.0 m/s, respectively. Determine the change of momentum of the ball. How much momentum is transferred to the wall?
7. A ball of putty of mass 0.25 kg strikes a wall at a speed of 5.0 m/s. How much energy is released in the process? What happens to the energy which is released? What processes is it used for?
8. A football player kicks a ball which then flies off at a speed of 9.0 m/s. The ball has a mass of 400 g. The foot touches the ball for approximately 0.10 s. a) Determine the average momentum current from the foot to the ball. b) Normally, the football will also rotate. Does this fact change your answer for the first question?
9. The system dynamics diagram in Fig.13.15 depicts a one-dimensional inelastic collision of two bodies. Take the bodies to have, respectively, masses of 2.0 kg and 1.0 kg, and speeds of 7.5 m/s and 0. Let the momentum current be constant at 50 N. a) How much momentum is exchanged? b) How long does the collision last? c) Determine the velocities as a function of time. d) Determine the rate at which energy is released as a function of time.
10. A car having a mass of 1000 kg moves in an easterly direction at an angle of 30° toward the south. Its speed is 30 m/s. a) Determine the components of the velocity and the momentum of the car in a coordinate system where the x-direction points east, and the y-direction north. b) What is the kinetic energy of the car? c) The coordinate system is rotated clockwise by an angle of 60°. How does the momentum vector change? d) By how much does the kinetic energy change after the coordinate system was rotated? e) Do the components of momentum and velocity change?
11. An elastic ball having a mass 0f 0.20 kg and a speed of 8.0 m/s hits a wall at an angle of 45°. Determine the motion of the ball after the collision. Determine what happens to the components of momentum which are parallel and perpendicular to the wall. What happens to the energy of the ball.
12. Earth and Moon orbit their center of mass in about 27 days. Determine the momentum of the system for an observer moving with its center of mass.
13. Electrons have a rest mass of $9.11 \cdot 10^{-31}$ kg. a) Determine their rest energy. b) If the energy of an electron is 1.2 times its rest energy, what is its mass? What is its momentum?

Problems

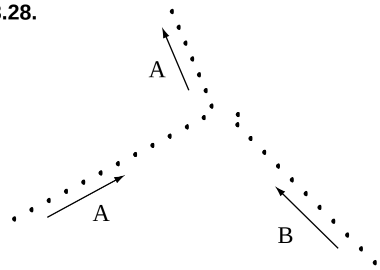
1. A flat open car having a mass of 1000 kg move horizontally without friction at a speed of 3.0 m/s. A person having a mass of 100 kg jumps off a tree into the moving vehicle. Then the person runs in the direction of motion and jumps off of the moving vehicle at a speed of 5.0 m/s relative to the vehicle. a) How large will the speed of the vehicle be after the person has jumped off? (Hint: Solve the problem in two steps.) b) How much energy was lost for motion after the person jumped onto the car? (Neglect the energy of vertical motion.) c) How much energy did the person release at least while gathering up speed for the jump off the car?
2. A ball of putty having a mass of 0.20 kg is thrown horizontally against a still standing block of wood (having a mass of 1.0 kg). The ball sticks to the block and the two bodies continue to move together at a speed of 4.0 m/s. a) Draw the fluid image of the process, and calculate the speed of the ball before the collision. b) Repeat the problem for an observer moving at a speed of 10.0 m/s in the same direction as the ball.
3. A bullet having a mass of 3.0 g and moving at a speed of 400 m/s first goes through a wooden board mounted on a glider on an air-track. Afterwards, it gets stuck in a board mounted on a second glider. The masses of the gliders (including boards) are 750 g and 747 g, respectively. They were at rest initially and collide after moving three and one meters, respectively. a) How fast do the gliders move after the bullet hit the boards? b) How much energy has been dissipated as the result of the bullet going through the first board?

FIGURE 13.27.



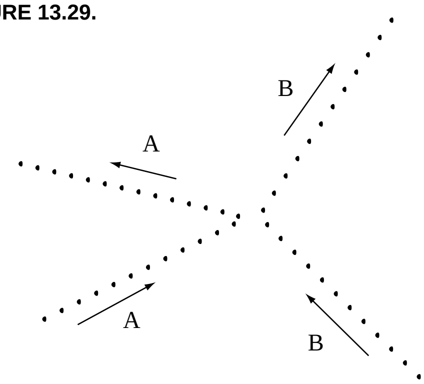
4. Determine the speed of the center of mass of the system of three bodies in Problem 3 before the bullet struck the first board, after it went through the first board, and after it got stuck in the second board. Interpret the result.
5. The collision in Fig.13.19 effectively lasts for about 0.10 s. Determine the direction and the magnitude of the average momentum flux during this period for bodies A and B. How does the direction of the momentum flux change if the collision is shorter?
6. Two discs move without friction on an air table and collide as shown in the stroboscopic image (Fig.13.28). The mass of the second disc is 1.5 times smaller than the mass of the first one. a) Determine the absolute value and the direction of the velocity of the second disc after the collision. b) Is the energy of the discs due to motion conserved during the collision? c) Determine the velocity vector of the center of mass of the system before and after the collision.

FIGURE 13.28.



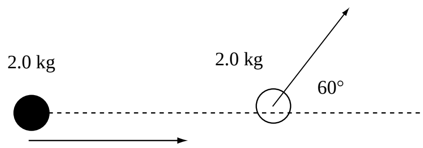
7. Two discs move without friction on an air table and collide as shown in the stroboscopic image (Fig.13.29). Determine the ratio of the masses of the two discs. Do we determine inertial or gravitational mass in this manner? Why?

FIGURE 13.29.



8. The black ball collides elastically at a speed of 10 m/s with the white ball which is at rest (Fig.13.30). The white ball moves at an angle of 60° to the original direction of the black ball. The speed of the white ball is measured to be 5.0 m/s. Both balls have a mass of 2.0 kg. a) In what direction and at what speed will the black ball be moving after the collision? b) If the collision takes 1/10 s, how large is the average momentum flux with respect to the black ball (absolute value and direction)?

FIGURE 13.30.



9. Electrons are accelerated through a voltage of 240 kV. Determine the energy and the speed of a single electron using the classical relation for kinetic energy. Since the speed of an electron is fairly close to the speed of light, determine the mass, momentum, and the speed according to the relativistic relations. Compare the two methods. The mass of an electron is $9.11 \cdot 10^{-31}$ kg.