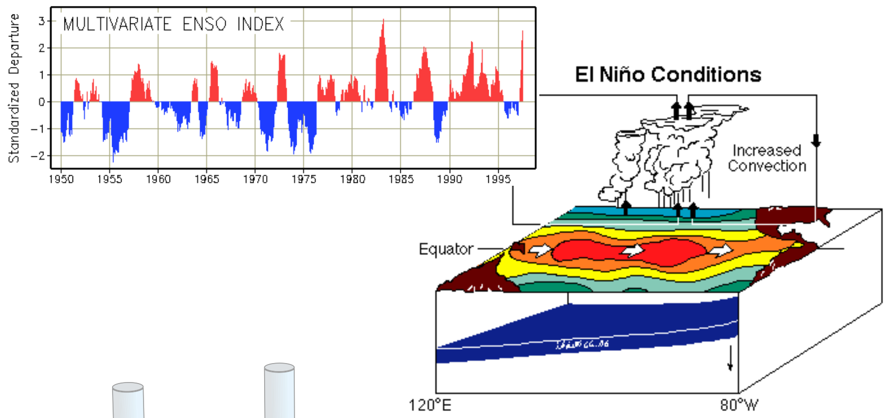
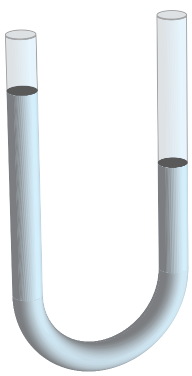
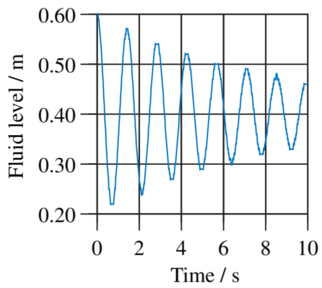
CHAPTER 3

Inductive Phenomena and Oscillations







CONTENTS

The El Niño Southern Oscillation (ENSO) is one of the large scale oscillatory phenomena on our planet. Oscillations can be observed in many simple and complex systems.

Many of the flows in nature exhibit what we might call inertia: changing them leads to observable effects. In general, it is not possible to turn flows on or off instantly; they take their time to change.

This behavior is called induction. Inductive phenomena abound in fluid flow, electricity, and motion. Here we will study the effect as it makes itself felt in fluid flow and electricity. First, we will see how including induction leads to more realistic models of dynamical processes. The the law of induction will be stated, and models of circuits including resistive and inductive elements will be investigated. We briefly take a look at two concrete inductive elements in hydraulics and electricity. Finally, we will investigate oscillations which arise in systems containing capacitors and inductors.

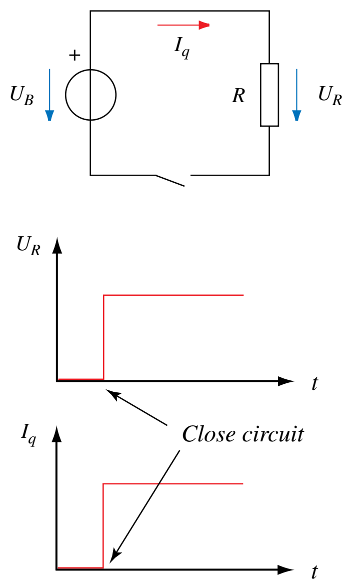


FIGURE 3.1. As the circuit is closed, the current jumps to its value calculated according to the resistance law.

3.1 Starting, Stopping, and Oscillating Currents

So far, our description of nature makes use of two phenomena: if fluids or electricity are stored in hydraulic or electric capacitors, the potential difference across the devices increases; and, if fluids or electricity flow through resistors, a drop of the associated potential occurs in the direction of flow. As we will see shortly, models of dynamical processes which only include capacitive and resistive behavior, cannot account for many important observations—namely those which have to do with starting and stopping currents, and with oscillatory behavior.

Currents in *RC* Systems and Real Behavior

Consider a simple electric circuit as in Fig.3.1 consisting of a battery, a resistor, and a switch. As the switch is closed, a current is set up which, according to Chapter 2, is given by

$$I_q(t) = \frac{U_R(t)}{R} \tag{3.1}$$

where $U_R(t)$ is the voltage across the resistor. If this quantity is constant, so is the current set up in the circuit. Note, that according to this model, the current attains the value calculated with the help of Eq.(3.1) as soon as the circuit is closed: I_q follows U_R instantaneously (Fig.3.1).

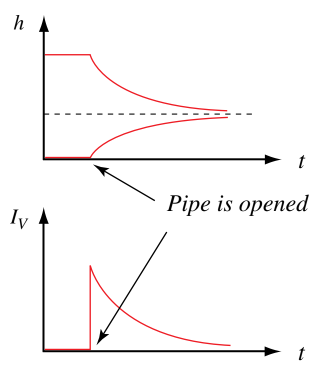


FIGURE 3.2. As the pipe connecting two fluid tanks is opened, the current of fluid suddenly jumps to its initial value.

As a second example study the model of two communicating fluid tanks described by the laws of capacitance and resistance. The pressure at the bottom of a fluid tank is determined by the level of the fluid, and the pressure difference from the outlet of one tank to the inlet of the next is responsible for the magnitude of the current of fluid through the connecting pipe, according to a relation analogous to Eq.(3.1).

As we know from the models studied in Chapter 1, the behavior of the system is described by the fluid levels in the two tanks as functions of time as in Fig.3.2. As soon as the pipe is opened, the fluid current must attain its largest value. It will decrease with time and reach a value of zero precisely when the levels of fluid in the tanks have become equal. This condition makes it impossible for oscillations of the fluid between the tanks to occur.

The sudden rise of a current from a value of zero to some other value required by the resistance law in Eq.(3.1) is certainly unrealistic. Moreover, we know that fluids can

Cover picture: Diagram explaining the origin of the El Niño phenomenon. The inset shows the summary of El Niño indicators. National Oceanographic and Atmospheric Administration.

oscillate between containers. A simple example of communicating vessels is a U-shaped glass pipe filled with mercury. If we raise the level of the fluid in one of the limbs and then let it flow, it will certainly oscillate for a while. Therefore, the models constructed from containers and resistors (*RC* systems) alone cannot explain what we observe in nature. We need to know how to describe the conditions which lead to changes in currents, or how changing currents induce other phenomena.

Induction and Inductive Driving Forces

Let us study the flow of a oil out of a tank having a straight horizontal pipe as in Fig.3.3. First hold the outlet (point *B*) of the pipe closed; the pressure of the fluid will be the same at the inlet and the outlet of the pipe, and there will be no current. If we now suddenly remove the lid from the outlet, the pressure of the fluid at *B* drops to ambient pressure, immediately leading to a pressure difference from *A* to *B*. The current will rise from zero to a its maximum in a certain amount of time. After that it will decrease just as the pressure difference decreases because there will be less oil in the tank as time passes.

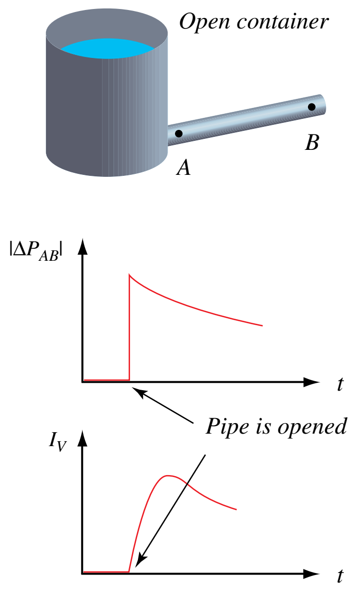


FIGURE 3.3. A fluid discharging through a long horizontal pipe. The current cannot jump suddenly to the value calculated according to the resistance law. Rather, it rises gradually.

The rise of the current from zero to its maximum value in a finite time span is new for us and cannot be explained in terms of capacitance and resistance. Observe that the current is zero right at the initial moment even though the pressure difference ΔP_{AB} is different from zero. Therefore, this pressure difference cannot be responsible for a current according to the law of resistance $I_V = -\Delta P_R/R_V$. Put differently, $\Delta P_R \neq \Delta P_{AB}$.

Rather, right at the beginning, $I_V = 0$, and therefore $\Delta P_R = 0$. This means that the entire pressure difference ΔP_{AB} must be the cause of a different phenomenon—namely the rise of the volume flux. Since the change of a current is related to what we have called inductive behavior, the pressure difference responsible for the rise of the current will be called the *inductive pressure difference* ΔP_L. We also call it the *inductive driving force*. At the first moment when the oil begins to flow we have $\Delta P_L = \Delta P_{AB}$.

The phenomenon of inductance is measured in terms of how fast the flux changes, i.e., in terms of the rate of change of the flux dI_V/dt. The rate of change of the current at the beginning can be represented by the slope of the I_V–t curve as it begins to rise from $I_V = 0$ (Fig.3.4). If the pressure difference available for increasing the current is the same, the current will rise faster or more slowly according to the properties of the system. If the amount of fluid is small, a given pressure difference ΔP_L will lead to a fast increase of I_V.

As time passes, the current increases. Since it is not zero any longer, there must be a pressure difference associated with the flow through the resistive element. In other words, the pressure difference ΔP_{AB} is not solely responsible for increasing the flux any longer. As the inductive pressure difference decreases, the current rises more slowly than before. This is exactly what we observe: the slope of the I_V–t curve decreases with time. We should expect the current to increase as long as it is smaller than $\Delta P_{AB}/R_V$.

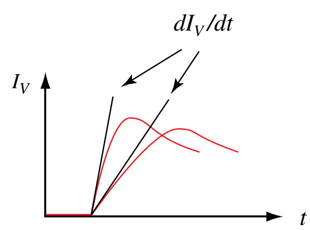


FIGURE 3.4. For a given inductive pressure difference, the current will rise faster if the amount of fluid in the pipe is small.

We can easily understand why there should be a pressure difference from *A* to *B* if the current of oil is supposed to increase with time. The fluid in the pipe has to be accelerated for this purpose. Having a higher pressure at *A* than at *B* accomplishes just that. (Remember that we also need a pressure difference if the flow is resistive; however, at the onset of the current, the flux is still zero, and resistance does not play a role yet.) The fluid is pushed more strongly from behind, and it accelerates, leading to an increase of the flux. On the other hand, a current decreasing with time must be connected to a positive inductive difference, meaning that the pressure increases in the direction of flow (again, if other effects such as fluid resistance do not alter our reasoning).

The cause of the inductive effect in electrical circuits is less obvious. It has to do with the magnetic field set up by the electric current. If the current changes, so does the magnetic field. Letting the magnetic field increase or decrease is related to a positive or negative voltage U_L, respectively (Sections 3.2 and 3.4).

Q

1. Why is the pressure difference in the direction of flow not due to fluid resistance if the flow just started?
2. The electric circuit in Fig.3.1 has just been closed. Why should U_R be zero, and why does this pose a problem?
3. Why does an *RC* model of a U-pipe filled with mercury not allow for oscillations of the fluid?
4. A pump produces a constant pressure difference ΔP_{AB} along a pipe containing a fluid. Initially, the fluid is at rest. Why should we expect the flux I_V to increase to the value of $\Delta P_{AB}/R_V$?
5. A *negative* inductive pressure difference is associated with an increasing current of fluid. Why is a *positive* inductive voltage responsible for increasing an electric current?

A

1. If the flow just starts, the volume flux is zero. Therefore, the resistive pressure difference $\Delta P_R = -R_V I_V$ must be zero as well.
2. The electric current should still be zero, meaning that $U_R = RI_q$ must be zero as well. According to the loop rule, the sum of all voltages in the circuit must add up to zero. With $U_B \neq 0$, and $U_R = 0$, we seem to encounter a contradiction which can only be resolved if we say that there is another voltage in the circuit, which must be responsible for the rate of change of the current.
3. In an *RC* system, the current has become zero when the fluid levels have become equal. Since this also means that the pressure difference is zero, the fluid cannot be driven to flow any longer.
4. When the current has reached that magnitude, the resistive pressure difference equals the pressure difference ΔP_{AB}. Therefore, there is no pressure difference left to continue accelerating the fluid.
5. A voltage is defined as the *negative* electric potential difference. Therefore, we have a *negative* electric potential difference associated with an *increasing* current.

3.2 The Law of Induction

Changes in currents are related to potential differences. The precise nature of this relationship will be studied here. We will begin with hydraulic circuits, and then use the analogy between hydraulics and electricity to formulate the law of induction for electric processes as well.

The Law of Induction in Hydraulics

The flow of fluids in straight pipes shows that—when a current is just beginning to flow—the rate of change of the flux I_V depends upon the pressure difference available for accelerating the fluid, which is the pressure difference ΔP_{AB}. In fact, experiments show that the initial slope of the rising I_V–t curve is proportional to the pressure difference across the pipe. Actually, the rate of change of I_V and the inductive pressure difference are always proportional, not just when a current starts. We only have to note that the inductive pressure difference is not equal to ΔP_{AB} if there is fluid friction.

Hydraulic induction is the result of the inertia of the fluid flowing through a pipe. Therefore, its should be possible to derive the relation between rates of change of currents and inductive pressure differences on the basis of the laws of mechanics. This we will do in Chapter 19. However, since the fundamental laws of motion cannot be proved either, we can just as well accept the observations as an expression of a law of hydraulics:

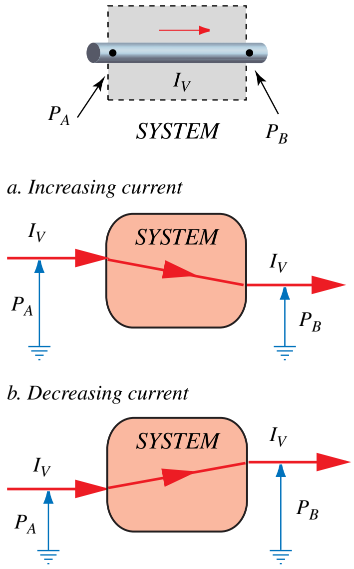


FIGURE 3.5. Increasing and decreasing currents are related to negative and positive inductive pressure differences, respectively.

■ *Law of induction: The inductive pressure difference ΔP_L and the rate of change of the flux of volume dI_V/dt are proportional. The constant of proportionality is called the inductance L_V of the system (fluid in a pipe):*

$$\Delta P_L = -L_V \frac{dI_V}{dt} \tag{3.2}$$

A current increasing with time is associated with a negative pressure difference, whereas a decreasing current leads to a positive ΔP_L (Fig.3.5).

EXAMPLE 3.1. Estimating the hydraulic inductance.

Water is at rest in a pipe. As the tap is opened, the pressure difference along the pipe is 2.0 bar. Initially, the current of water increases at a rate of 2.0 liters/s^2. a) How large is the inductance of the water in the pipe? b) How fast would the current rise initially if the pipe were twice as long?

SOLUTION: a) The inductance is calculated from Eq.(3.2):

$$L_V = -\frac{\Delta P_L}{dI_V/dt} = -\frac{-2.0 \cdot 10^5\,\text{Pa}}{2.0 \cdot 10^{-3}\,\text{m}^3/\text{s}^2} = 10^8\,\frac{\text{Pa}\cdot\text{s}^2}{\text{m}^3}$$

This figure is fairly common for fluid pipes. An oil pipeline of several 100 km length has a similar hydraulic inductance.

b) If all other factors are kept constant, doubling the length of the pipe will double the amount of water to be accelerated. Therefore we expect the inductance to increase by a factor of 2. This will lead to an initial rate of change of the flux of 1.0 liters/s^2.

■

Pressure Differences in Hydraulic Circuits

It is important to understand the nature of a pressure difference across a pipe containing a fluid (Fig.3.6). As we have seen, the pressure difference from A to B resulting, for example, from a pump, may be related to more than just one phenomenon. The pressure may change in the direction of flow due to fluid friction, but it may just as well decrease or increase from A to B because the current is changing in time. We know two extremes: either, the current is not changing and the entire pressure difference relates to fluid friction, or the current is zero at a moment in which case the pressure difference is solely due to the time rate of change of the current. In other words, in these extreme cases we have, respectively, $\Delta P_{AB} = \Delta P_R$, or $\Delta P_{AB} = \Delta P_L$. In general, the pressure difference is due to both phenomena simultaneously, which means that it is divided between the processes:

$$\Delta P_{AB} = \Delta P_L + \Delta P_R \tag{3.3}$$

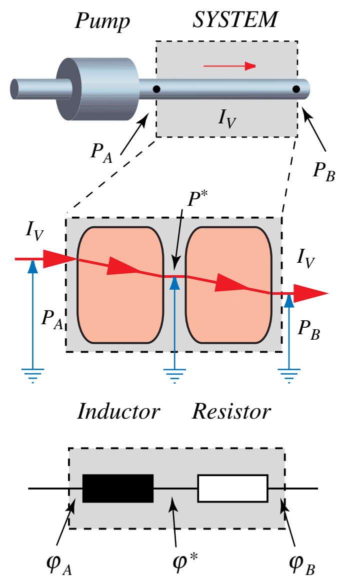


FIGURE 3.6. The fluid in a pipe may be subject to both resistance and inductance. While the phenomena share the space from A to B, we model them as happening consecutively. Together, ΔP_L and ΔP_R make up ΔP_{AB}.

(This can be visualized more easily in an electric circuit (Fig.3.6) where inductors and resistors are separate elements placed one after another, i.e., in series. See the next section.) We may imagine the pressure to drop (or rise) first because of the inductive process, and then to drop due to friction. Together the inductive and resistive pressure differences are equal to the pressure difference from A to B.

Remember that ΔP_R may only be negative (in the direction of flow), whereas ΔP_L may, as shown in Eq.(3.3), take either sign, making ΔP_{AB} either negative or positive. A positive value of ΔP_L means that the fluid flux is decreasing with time.

EXAMPLE 3.2. Resistive and inductive pressure differences.

Water is flowing out of a slender tank through a straight horizontal pipe as in Fig.3.3. The pipe has a length of 1.75 m, and a radius of 2.0 mm. Initially, the level of water in the tank is 0.20 m. The graphs in Fig.3.7 show the modeled behavior of the system with and without taking inductance into account. The flow is laminar at all times. a) Estimate the inductance and the resistance of the fluid in the pipe. b) Estimate the inductive and resistive pressure differences for $t = 20$ s. c) Determine the function $I_V(t)$ after a fairly long time ($t > 10$ s), and calculate the cross section of the tank. d) Determine the total pressure difference along the pipe at $t = 20$ s, and estimate the volume of water in the tank at that moment.

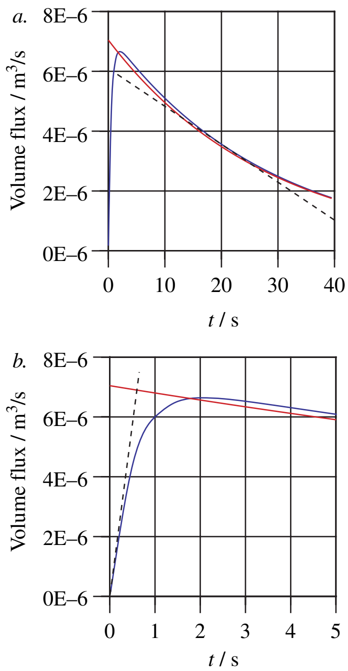


FIGURE 3.7. Graphs showing the volume flux in a hydraulic system as a function of time. The line starting at a value of 7E–6 units is the computed result without taking induction into account. Figure b shows an enlargement of the upper graph.

SOLUTION: a) As in Example 3.1, the inductance can be computed if we know the inductive pressure difference and the rate of change of the flux of volume. The latter may be obtained graphically from the slope of the curve rising at $t = 0$ (Fig.3.7b). Graphical determination yields a value of $6 \cdot 10^{-6}/0.45\ \mathrm{m^3/s^2} = 1.3 \cdot 10^{-5}\ \mathrm{m^3/s^2}$. The inductive pressure difference at $t = 0$ is equal to the pressure difference across the fluid column in the tank:

$$\Delta P_L(t=0) = -\rho g h(t=0) = -1000 \cdot 9.81 \cdot 0.2\,\mathrm{Pa} = -1.96 \cdot 10^3\,\mathrm{Pa}$$

The inductance turns out to be

$$L_V = -\frac{\Delta P_L}{dI_V/dt} = -\frac{-1962\,\mathrm{Pa}}{1.3 \cdot 10^{-5}\,\mathrm{m^3/s^2}} = 1.5 \cdot 10^8\,\frac{\mathrm{Pa \cdot s^2}}{\mathrm{m^3}}$$

The resistance in laminar flow is

$$R_V = \frac{8\eta l}{\pi r^4} = \frac{8 \cdot 0.0010 \cdot 1.75}{3.14 \cdot 0.0020^4} = 2.79 \cdot 10^8\,\mathrm{Pa \cdot s/m^3}$$

b) The inductive pressure difference is obtained with the help of the law of induction if we know the rate of change of the current. This quantity can be estimated graphically from the first graph (Fig.3.7a) which yields dI_V/dt ($t = 20$ s) $\approx -3.7 \cdot 10^{-6}\ \mathrm{m^3/s}$ / 30 s $= -1.2 \cdot 10^{-7}\ \mathrm{m^3/s^2}$. Therefore

$$\Delta P_L = -L_V\, dI_V/dt \approx -1.5 \cdot 10^8 \left(-1.2 \cdot 10^{-7}\right)\mathrm{Pa} = 18\,\mathrm{Pa}$$

The resistive pressure difference is calculated on the basis of the resistance law

$$\Delta P_R = -R_V I_V = -2.79 \cdot 10^8 \cdot 3.6 \cdot 10^{-6}\,\mathrm{Pa} = -1.00 \cdot 10^3\,\mathrm{Pa}$$

The current at $t = 20$ s is obtained from the first graph (Fig.3.7a).

c) After the current has risen quickly, the inductive phase is mostly over, and inductive pressure differences become small. The function $I_V(t)$ is almost identical to the one calculated without taking induction into account. Therefore we can determine an approximation to $I_V(t)$ by calculating the solution for discharging a container, looking at the system as a combination of capacitor and resistor:

$$I_V(t) = I_{Vo} \exp(-t/\tau) \quad , \quad \tau = R_V K_V \quad , \quad I_{Vo} = -\Delta P_{AB}(0)/R_V$$

We have $I_{Vo} = 7.0 \cdot 10^{-6}$ m^3/s. The current will decrease to 1/e of the initial value in one time constant τ. Inspection of the first graph (Fig.3.7a) shows that $\tau \approx 30$ s. This yields $K_V = \tau/R_V = 1.08 \cdot 10^{-7}$ m^3/Pa. From this we can calculate the cross section of the container:

$$A = \rho g K_V = 1.05 \cdot 10^{-3}\,\mathrm{m}^2$$

d) The total pressure difference is the sum of the inductive and resistive differences:

$$\Delta P_{AB} = \Delta P_L + \Delta P_R = 18\,\mathrm{Pa} + (-1000\,\mathrm{Pa}) \approx -980\,\mathrm{Pa}$$

Therefore, the water level is 0.10 m. The volume of water can be calculated with the help of the capacitance which we get from the time constant $\tau = R_V K_V$; therefore, $K_V = \tau / R_V = 30/2.8 \cdot 10^8$ m^3/Pa = $1.1 \cdot 10^{-7}$ m^3/Pa. Now we have

$$V(20) = K_V \Delta P_K(20) = K_V\left(-\Delta P_{AB}(20)\right) = 1.1 \cdot 10^{-7} \cdot 980\,\mathrm{m}^3 = 1.06 \cdot 10^{-4}\,\mathrm{m}^3$$

Naturally, we could also determine the amount of water which has left the tank in the time span from 0 s to 20 s (by determining the area between the I_V–t curve and the t–axis), and deduct the value from the initial volume.

■

Induction in Electricity

Electricity and hydraulics share deep analogies. Both phenomena are described in terms of a substancelike quantity which we can imagine to flow and to be stored in systems. Electric charge and amounts of fluids each satisfy a law of balance. Each quantity possesses a potential, and potential differences add up to zero along closed loops in circuits. Differences of the potential related to the processes are responsible for resistive flow, and in both cases we introduce capacitive and resistive laws.

The law of induction. The analogy carries over to inductive phenomena (Table 3.1). It is equally impossible to have electric and hydraulic currents suddenly jump to values determined solely by resistive laws, and there are electric circuits which admit oscillatory behavior. These phenomena are said to be the result of inductive elements combining with those we already have used in the description of electric circuits. In inductive elements, the rate of change of the electric current is related to an inductive voltage according to the law of induction:

■ *Law of induction: The inductive voltage U_L and the rate of change of the electric current dI_q/dt are proportional. The constant of proportionality is called the inductance L of the element (SI-unit: Henry or H):*

$$U_L = L\frac{dI_q}{dt} \tag{3.4}$$

A current increasing with time is associated with a positive inductive voltage, whereas a decreasing current leads to a negative U_L.

Magnetic fields. The fundamental difference between electric and hydraulic induction can be found in the cause of the phenomenon. Fluids need a pressure difference to change their currents, since they are subject to inertia. Electric inductance, however, is caused by the coupling of electric currents and magnetic fields (Part V). Electric currents produce magnetic fields whose strength depends upon the current. If we let a current be increased, the magnetic field associated with it must increase as well. The

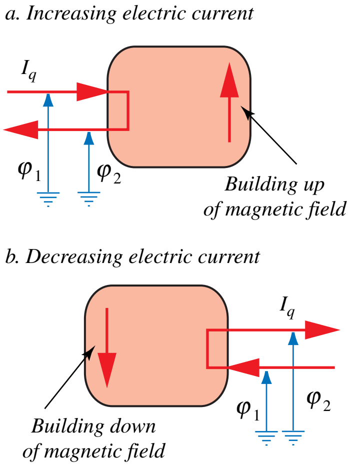


FIGURE 3.8. If an electric current increases, it induced a potential difference which makes it flow downhill, causing a magnetic field to be increased. A decreasing current is related to a decreasing field.

phenomenon of the increase of the field must be caused by the electric current. Electricity causes other phenomena if it flows either downhill or uphill: this is the source of the electric potential difference related to inductive processes (Fig.3.8).

TABLE 3.1. **Electric and hydraulic elements compared**

	Capacitors	Resistors	Inductors
Hydraulics	$\dot{V} = K_V \dot{P}$ $\Delta V = K_V \Delta P$	$I_V = -\Delta P_R / R_V$	$\Delta P_L = -L_V \, dI_V / dt$
Electricity	$\dot{q} = C\dot{U}_C$ $q = CU_C$	$I_q = U_R / R$	$U_L = L dI_q / dt$

Voltages in electric circuits. Basically, all elements used in electric circuits—such as wires, resistors, and capacitors—share the property of being inductors as well. However, normally, their inductance is very small allowing us to treat the elements as having just one property. To get large inductances, inductors are made from wire wound to different types of coils, such as a straight solenoid (Section 3.4). In circuit diagrams, inductors are represented as black rectangles (Fig.3.9).

If an electric element in a circuit has properties of both inductance and resistance, the voltage across the element is the sum of the inductive and resistive voltages (Fig.3.9):

$$U_{AB} = U_L + U_R \tag{3.5}$$

This relation is analogous to what we have seen in the case of hydraulic circuits, as expressed by Eq.(3.3).

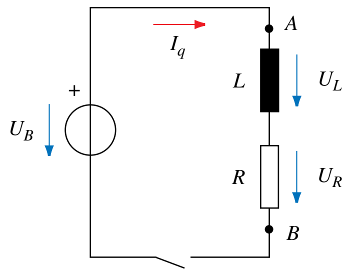


FIGURE 3.9. Inductive and resistive elements in electric circuit diagrams. In models, inductors and resistors are represented as just that, even though real elements—such as long wires wound to a solenoid—have both properties combined.

Q

1. Why is there no minus sign in the law of electric induction in Eq.(3.4) as there is in the hydraulic case in Eq.(3.2)?
2. Consider an electric circuit as in Fig.3.9 or a pipe containing a fluid as in Fig.3.6. Is it possible to have a negative voltage or a positive pressure difference from *A* to *B*?
3. In a circuit such as in Fig.3.9, the switch is closed. Why does the electric current attain a steady value? How is it determined?

A

1. The electric law of inductance is written in terms of the voltage rather than the potential difference. Therefore, the sign changes.
2. The condition may arise if a current is decreasing fast leading to a large negative inductive voltage or a large positive inductive pressure difference.
3. The current increases with time, letting the potential differences across the resistor increase. As $U_R = -U_B$, we have $U_L = 0$ V, and the current cannot change any longer. The value of the current is determined from the resistance law using $U_R = -U_B$.

3.3 *LR* Models of Hydraulic and Electric Circuits

The simplest systems including inductive elements are those represented in Figs.3.6 and 3.9. A pump or a battery are used to set up a constant pressure difference and a constant voltage, respectively, from point A to point B. What are the system dynamics models and the system behavior?

Starting and Stopping a Current in an *LR* Circuit

Let us first discuss the expected behavior which should be the same in both cases. Initially, right after the circuits have been closed, the potential difference from A to B is equal to the negative value of the pressure difference of the pump and the voltage of the battery, respectively. Since the current is zero at this time, resistive potential differences are zero as well. Therefore the current should change at a rate $dI/dt = -\Delta\varphi_{AB}/L$ (Fig.3.10). As the current increases, the magnitude of the (negative) resistive potential difference must increase as well. The magnitude of the inductive potential difference must therefore decrease with time, leading to a decreasing slope of the rising current. Finally, when the current reaches a value of $I = \Delta\varphi_{AB}/R$, the inductive potential difference is zero, and the current will not change any longer. Naturally, the process can be reversed: turning a current off does not lead to an immediate result; rather, the current decays exponentially.

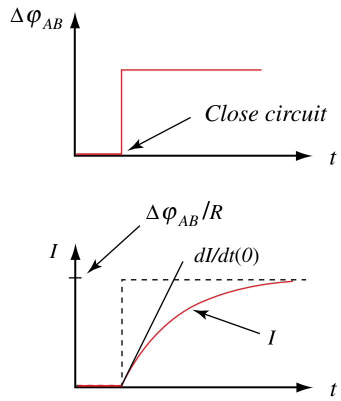


FIGURE 3.10. Simple *LR* behavior: the current rises exponentially and approaches a steady value.

The behavior described here in qualitative terms can be easily observed, for example, by measuring and displaying the quantities in an electric circuit with the help of a cathode ray tube (Fig.3.11).

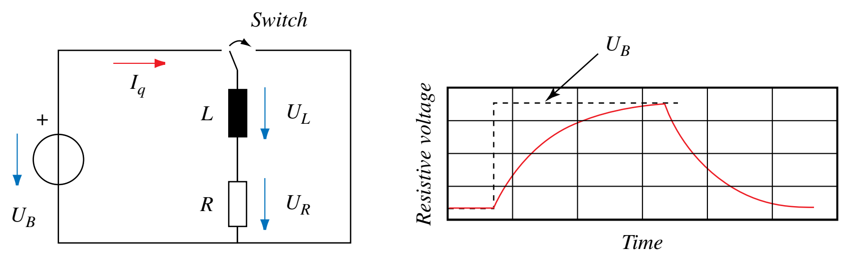


FIGURE 3.11. Turning an electric current on and off. The diagram shows the voltage measured across the resistor.

Inductive Time Constant

As we shall see below, the current is an exponential function of time. We have encountered such behavior before (Chapters 1 and 2). In Chapter 2 we introduced the notion of the (capacitive) time constant. Obviously, the behavior over time of *LR* systems can be described in similar terms. We introduce the *inductive time constant* τ_L which—as we have seen in Chapter 2—is measured by the section determined by the tangent to the curve at the moment the circuit is closed, and the horizontal line representing the steady value of the current (Fig.3.12). The graphical representation also tells us how this time constant can be determined. The rate of change of $I(t)$ at the initial moment is equal to the ratio of the steady value of the current reached and the time constant:

$$dI/dt = \Delta\varphi_{AB}/R/\tau_L$$

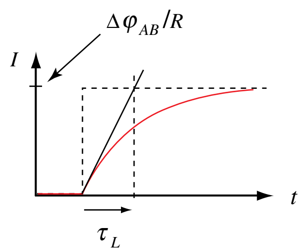


FIGURE 3.12. Graphical definition of the inductive time constant.

Since the time rate of change of the current is equal to the ratio of the initial value of the potential difference and the inductance, we have

$$\tau_L = L/R \tag{3.6}$$

The time constant also represents the time span necessary for the current to rise to a fraction of 1 – 1/e ≈ 0.64 of the final steady value.

A System Dynamics Model of Starting a Current

Consider the electric circuit of Fig.3.9. As we close the switch, the current starts rising from a value of zero to reach a final steady value. How the current changes is described by the law of induction: the rate of change of the current is equal to the ratio of the inductive voltage and the inductance of the inductive element. This is our starting point to construct the system dynamics diagram and model. We draw symbols representing the inductive voltage U_L, the inductance L, and the rate of change of the current, and connect the former two elements to the latter (Fig.3.13). This is used to represent the equation

$$rate\ of\ change\ of\ I_q = U_L / L$$

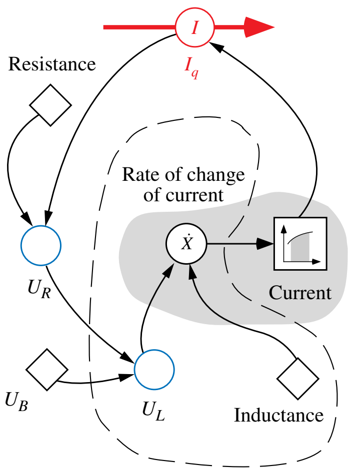


FIGURE 3.13. System dynamics diagram of the model of an *LR* system representing the starting of a current. Note the law of induction (dashed line), and the process of integrating the rate of change of the current (gray underlined area). The latter adds an automatic numerical process to the system model.

The next step involves obtaining the current from its rate of change. There is a new aspect in the model we are constructing: we have not yet encountered laws stating the rate of change of a quantity, except—indirectly—laws of balance where the sum of all currents with respect to a system is equal to the rate of change of the quantity stored. Here, however, we are not dealing with a law of balance. Still, on a purely mathematical level, there is no difference between what we need to do—namely calculate the current from its rate of change—and calculating the system content from the sum of the currents. Both processes require integrating a rate of change to obtain the quantity we are looking for. Since we do not deal with a law of balance here, the system dynamics diagram should contain a new symbol denoting the operation

$$current = \int \left(rate\ of\ change\ of\ I_q \right) dt$$

(Fig.3.13). To obtain the current as a function of time we need its initial value which, in our example, is zero. A system dynamics program obtains the solution of this relation by the same numerical method which is used in the case of laws of balance.

Now we are ready to proceed. Knowing the current, we can calculate the resistive voltage from the resistance law, and using Eq.(3.5) we obtain the inductive voltage. Note that U_{AB} in Fig.3.9 is the negative value of U_B (voltage of the battery). This closes the feedback loop present in the simple *LR* system (Fig.3.13).

Formal Description of *LR* Systems

Filling in the equations representing the relations in the system dynamics diagram of Fig.3.13 finally yields the complete mathematical model. The equations making up this model are:

$$\begin{aligned} \frac{dI_q}{dt} &= U_L / L \quad , \qquad I_q(0) = 0 \\ U_L &= -U_B - U_R \\ U_R &= R I_q \end{aligned} \tag{3.7}$$

The model in Eq.(3.7) is made up of three equations and an initial value. The first equation takes the form of a differential equation for $I_q(t)$. The second determines the inductive voltage from the other voltages, and the last yields the resistive voltage from the resistance law. Together they determine the solution of the problem. Inserting the latter two equations into the former we have

$$\frac{dI_q}{dt} = \frac{1}{L}\left(-U_B - RI_q\right) \quad , \quad I_q(0) = 0$$

Its solution is

$$I_q(t) = -\frac{U_B}{R}\left(1 - \exp\left(-t/\tau_L\right)\right) \tag{3.8}$$

which can be demonstrated by plugging the result back into the differential equation. Here, τ_L is the inductive time constant already expressed in Eq.(3.6). Note that since U_B is negative, the current is a positive quantity increasing from a value of zero and reaching a steady value after long time.

EXAMPLE 3.3. Inductive time constants.

In Example 3.2, a) determine the inductive time constant by graphical means. b) Compare the result to a determination with the help of Eq.(3.6). c) If the fluid level in the tank could be kept constant, how long would it take for the current to reach 99% of its final value?

SOLUTION: a) The current rises to about 64% of its final value in one time constant. Since there is no steady state, we use the maximum value of the current in the graph of Fig.3.7a, which is about $6.7 \cdot 10^{-6}$ m^3/s. 64% of this ($\approx 4.3 \cdot 10^{-6}$ m^3/s) is reached at roughly 0.5 s after the start. A rough measure of the time constant therefore is 0.5 s.

b) On the other hand, the inductive time constant is

$$\tau_L = \frac{L_V}{R_V} \approx \frac{1.5 \cdot 10^8}{8 \cdot 0.0010 \cdot 1.75 / \left(\pi 0.0020^4\right)} \mathrm{s} = 0.54\,\mathrm{s}$$

c) With a constant fluid level in the container, the driving pressure difference stays constant. In this case we have the simple problem of starting a current discussed above. Since $-U_B/R$ is the initial current, we have

$$0.99 = 1 - \exp\left(-t/\tau_L\right)$$

This can be solved for the unknown time span:

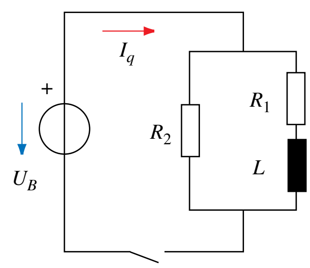
$$t = -\tau_L \ln(1 - 0.99)$$

With a time constant of 0.5 s, this turns out to be 2.3 s.

■

EXAMPLE 3.4. An electric circuit containing an inductor.

Consider the electric circuit shown in the diagram. An ideal inductor is connected in series to a resistor, having a resistance of 24 Ω, and both are connected in parallel to another resistor (72 Ω). Together the three elements are connected to a battery of constant voltage (12 V). At $t = 0$ s the circuit is closed. a) Give a qualitative sketch of the inductive voltage, i.e. the voltage across the solenoid, as a function of time. Determine the maximum value of the voltage. b) Give a qualitative sketch of the current through the first of the resistors. What is the maximum value of the current? c) Sketch the current through the battery as a function of time.



SOLUTION: As the circuit is closed at $t = 0$ s, the voltage across R_2, and across R_1 and the solenoid, jumps to 12 V. In the branch with the inductor, the current is zero since it has not had time to rise. Since there is no inductor in the branch with R_2, the current may jump to an appropriate value instantaneously. In the section with the solenoid, the current will rise according to the rules discussed in this section on *LR* circuits; as depicted in Fig.3.12, it rises until it reaches a steady value.

a) Since there is no current through the branch with the first resistor and the inductor initially, the voltage across R_1 must be zero. Therefore, at $t = 0$ s, the voltage $U_L = 12$ V, which is the largest value. From there it decreases exponentially to zero when steady conditions are reached (Fig.3.15a).

b) As discussed, the current through R_1 is zero initially. When steady conditions have been attained, the voltage across the inductor is zero, which means that $U_{R1} = 12$ V. As a consequence the largest value of the current is 12 V / 24 Ω = 0.50 A (Fig.3.15b).

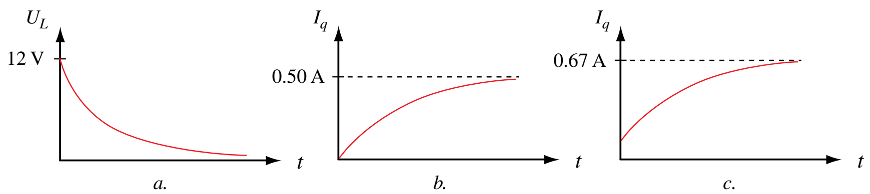


FIGURE 3.15. Voltage and currents as functions of time.

c) The current through the branch with R_2 obeys the resistance law. Since the voltage is a constant 12 V, the current is 12 V / 72 Ω = 1/6 A. We have to add this constant current to the one through R_1 to obtain the current through the battery (Fig.3.15c).

■

3.4 Inductive Elements

The phenomenon of induction has to do with inertia of fluids and magnetic fields in hydraulics and electricity, respectively. These observations permit us to relate inductances of elements to mechanical properties of fluids in the case of hydraulics, and properties of electromagnetic fields in the case of electricity. However, to do so we need theories of mechanics and electromagnetism. Since these will be studied later (Part III and Part V), we will present only a couple of results of practical interest at this point. These are the inductances of a fluid in a straight pipe, and of a long wire wound to a solenoid.

Hydraulic Inductance of a Fluid in a Pipe

The inductance of a fluid in a pipe of length l and radius r depends upon the density of the fluid ρ, the length l, and the cross section $A = \pi r^2$ of the pipe. It increases with the density and the length, and decreases with increasing cross section:

$$L_V = \frac{\rho l}{A} \tag{3.9}$$

While the first two relations are easy to understand, the latter is more difficult to determine. First, the inertia, i.e. the mass of the fluid to be accelerated, increases linearly with density and length. Intuitively, it makes sense that twice the inductive pressure difference leads to the same rate of change of the current of fluid if either the length or density are doubled. Figuring out the dependence of the inductance upon the cross sectional area of the pipe, however, causes more problems. On the one hand, twice the cross section gives twice the surface for the pressure to act; on the other hand twice as much fluid has to be accelerated. These factors cancel each other which means that the same inductive pressure difference leads to the same speeding up of the fluid. However, twice the cross section means half the fluid speed for a given volume current. Therefore, twice the cross section leads to twice the rate of change of the current for a given inductive pressure difference.

There is an important point to be noted about hydraulic inductors. We know that inductance is related to the rate of change of the current of fluid through a pipe or similar element. Having this rate of change, we have an induced pressure difference. It is possible, however, to have pressure differences in frictionless flow do to still another phenomenon. If a steady current flows through a narrowing pipe, its pressure changes in the direction of flow (it decreases; see Chapter 22). Since the current does not change, the pressure difference must be the result of a phenomenon which is different from hydraulic inductance. Therefore, here we will consider only pipes having constant cross section.

Electric Inductance of a Solenoid

A solenoid is a tightly wound straight coil of wire. With an electric current flowing through the wire, a magnetic field is set up in the cylindric space. Coils of this type are often used for electromagnets. The magnetic field can be made even stronger if the solenoid is filled with a core composed of ferromagnetic materials such as iron.

As we shall see in Part V of the book, the magnetic field depends upon the magnitude of the current flowing through the wire, and the number of windings per length of the coil. Since the increase or decrease of the field occupying the space inside the solenoid is responsible for the inductive behavior of the element, the inductance is expected to depend upon its volume and the number of windings per unit length. In Chapter 27, we will show that

$$L = \mu_o n^2 A l \tag{3.10}$$

Here, μ_o is a fundamental constant, called the permeability constant having a value of $4\pi \cdot 10^{-7}$ H/m. The number of windings per meter is abbreviated by n, and A and l are the cross section and the length of the solenoid, respectively. With an iron core, the field and the inductance of the coil increase several hundred or thousand fold.

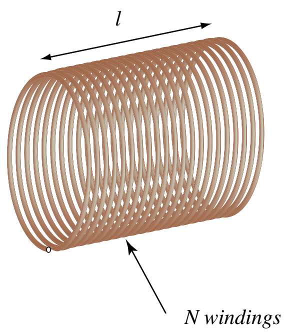


FIGURE 3.16. A solenoid, a straight tightly wound coil of wire, produces a magnetic field filling the interior, directed along the axis of the coil.

EXAMPLE 3.5. Inductance of water pipes.

Pipes having a diameter of 0.60 m lead from an artificial lake down into the valley. Their length is 350 m. a) What is the hydraulic inductance of a single pipe? b) Water is flowing through a pipe with a current of 1.0 m^3/s. If the current had to be stopped in 0.10 s, how large would the induced pressure difference be?

SOLUTION: a) The hydraulic inductance of a pipe is determined according to Eq.(3.9):

$$L_V = \frac{\rho l}{A} = \frac{\rho l}{\pi r^2} = \frac{1000 \cdot 350}{\pi \cdot 0.30^2} \frac{\mathrm{Pa \cdot s^2}}{\mathrm{m^3}} = 1.24 \cdot 10^6 \frac{\mathrm{Pa \cdot s^2}}{\mathrm{m^3}}$$

b) Because of the change of the current of water, there will be an induced pressure difference along the pipe. With an average rate of change of the current of 10 m^3/s^2, we have

$$\Delta P_L = -L_V \frac{dI_V}{dt} = -1.24 \cdot 10^6 (-10) \mathrm{Pa} = 124 \mathrm{bar}$$

■

EXAMPLE 3.6. Inductance of solenoids in series.

A long wire is wound up in series on two cardboard pipes. The first has a length of 20 cm and a diameter of 6.0 cm, while the length and the diameter of the second are 10 cm and 4.0 cm, respectively. On the first cardboard pipe, there are 200 windings, on the second there are 300. Determine the inductance of the setup.

SOLUTION: First, we calculate the individual inductances according to Eq.(3.10). They are

$$L_1 = \mu_o n_1^2 A_1 l_1 = 4\pi 10^{-7}\left(\frac{200}{0.2}\right)^2 \pi 0.030^2 \cdot 0.20\,\mathrm{H} = 0.71\,\mathrm{mH}$$

$$L_2 = \mu_o n_2^2 A_2 l_2 = 4\pi 10^{-7}\left(\frac{300}{0.1}\right)^2 \pi 0.020^2 \cdot 0.10\,\mathrm{H} = 1.42\,\mathrm{mH}$$

Now we have to determine the inductance of two inductors in series. The voltage across both together is the sum of the individual voltages:

$$U_L = U_{L1} + U_{L2}$$

Using the law of induction in Eq.(3.4), we obtain

$$U_L = L_1\frac{dI}{dt} + L_2\frac{dI}{dt} = \left(L_1 + L_2\right)\frac{dI}{dt}$$

since the current and its rate of change are the same for both elements. We can conclude that the inductance of inductors in series is the sum of the individual inductances. In our case:

$$L = L_1 + L_2 = 2.13\,\mathrm{mH}$$

■

3.5 Oscillatory Systems

Oscillatory processes—simple and complex—abound in nature, and in technical and social systems. They are by no means just a physical phenomenon. However, physical oscillatory systems are fairly easy to study, and they demonstrate the structure hidden behind the appearance.

Put simply, oscillations are repetitive phenomena. Water swapping from one tank into another, and back, is an example of a simple oscillatory system (Fig.3.17). Here, the

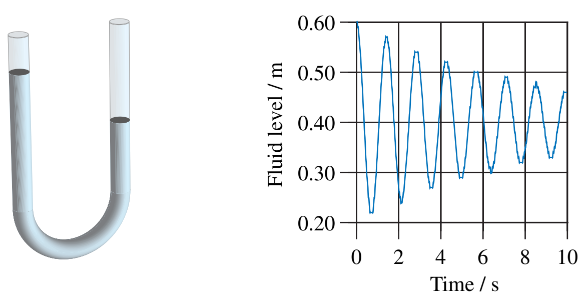


FIGURE 3.17. Many systems exhibit oscillatory behavior—some simple like mercury in a U-tube…

oscillations are periodic, usually decreasing with time, showing a rather simple form. There are more complex phenomena, such as oscillations in chemical systems. If reactions occurring in a reactor are fed by a flow of some of the species—or if some species are removed—and if the reactions satisfy certain rules, the amount of the species involved may oscillate back and forth. The calcium oscillation shown in Fig.3.18 is periodic like the one of water shown above, but the shape of the curves is rather different from that of a simple oscillation.

Many natural systems exhibit very complex, even aperiodic changes. Still, the basic characteristic of oscillatory change is clearly visible. Consider the El Niño Southern Oscillation, a phenomenon in which the coupling of the atmosphere and the ocean over the equatorial Pacific creates a large scale phenomenon of meteorological change which repeats itself every two to seven years or so (Fig.3.19).

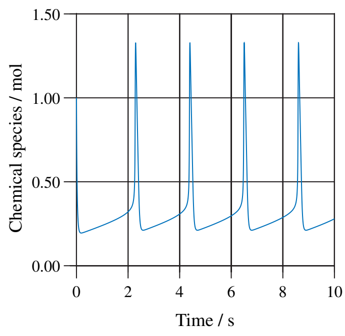


FIGURE 3.18. …some more complex like this chemical system (a calcium oscillator)…

During non-El Niño conditions (Fig.3.19a), trade winds blow towards the west across the equatorial Pacific. Warm surface water accumulates in the west Pacific, where sea surface temperatures are about 8°C higher than off the coast of South America; the lower temperatures are due to an upwelling of cold water from deeper levels. The cool water is within 50 m of the surface near the American continent. During an El Niño (Fig.3.19b), the trade winds let off in the central and western Pacific. This leads to a depression of the boundary between warmer and colder water in the eastern Pacific, and an elevation near Asia. The surface of colder water dropps to about 150 m off the coast of South America. The efficiency of upwelling to cool the surface is cut off and the sea surface temperatures rise. The changes in ocean currents is reflected in a change of the weather as well. In particular, the easterly trade winds are weakened during El Niño, and rainfall shifts eastward following the warm water.

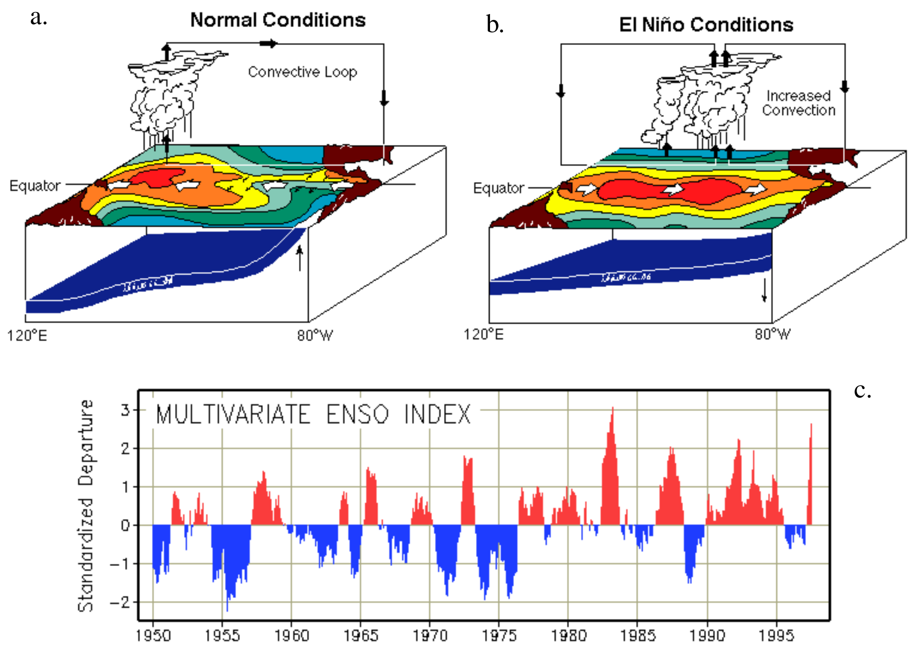


FIGURE 3.19. …and some very complex. The El Niño Southern Oscillation (ENSO) is an oscillatory phenomenon involving the atmosphere and the ocean in the region of the equatorial Pacific. The ENSO index (Figure c) is a combination of different measures which indicate presence or absence of the El Niño effect.

Here, we will first study oscillations from a more qualitative viewpoint, describing the behavior, discussing the example of hydraulic oscillations, and creating a first system dynamics diagram. In Section 3.6, we will turn to the mathematical description of the phenomenon.

Communicating Containers

Why do some systems show oscillatory behavior? To answer this question, let us study two communicating water tanks (Fig.3.20). We know that if we fill them with a highly viscous oil up to different levels and then watch the fluid flow, we will see a behavior as in Fig.3.2. The current through the connecting pipe will decrease with time from its maximum value, and the fluid in the tanks will reach the same levels and then stop flowing. We know from the models constructed in Chapters 1 and 2 that this type of behavior is the result of the interplay of storage devices (capacitors) and resistors. We speak of so-called *RC*-systems.

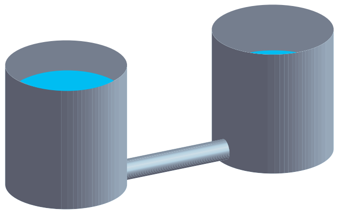


FIGURE 3.20. Water oscillates between two communicating tanks.

Fluids oscillating back and forth: the role of inertia. If, on the other hand, a fluid with low viscosity is used, and if the pipe is made wide enough, we will encounter a totally different phenomenon. Let the fluid be higher in the left tank in Fig.3.20. First, as we already know from Section 3.1, the current will rise to a certain value from zero, rather than start at the highest rate. The fluid begins to flow gradually from the left to the right tank. Moreover, experience tells us that the current will be largest—rather than zero—when the fluid levels have reached the same value. Since the fluid is still rushing though the pipe, the level in the right tank will continue to rise, and it will decrease further in the left container. At some point, the fluid level in the right tank will reach its highest value, and the current will momentarily stop. Now the flow begins to reverse its path, until the left tank is filled to its highest point again. In this manner the fluid oscillates back and forth between the tanks. With fluid friction present, we expect the amplitude of the oscillation to decrease with time as shown in the graph of Fig.3.17.

What makes the fluid current rise gradually at the beginning and reach its maximum value when the fluid levels are equal, instead of stopping at that point? We already know the answer from Section 3.1: induction is the cause of this phenomenon. Therefore we should expect systems of containers and pipes which include the effects of induction to exhibit oscillatory behavior. Indeed, this is the case. In summary, for a system to show oscillatory behavior two conditions have to be fulfilled:

- there must be two storage devices for the quantity of fluid to be able to flow back and forth, and
- the flow must show inductive behavior so that it will not stop precisely the first time the fluid levels have become equal.

System dynamics diagrams. Let us construct the system dynamics diagram of the system of communicating containers in Fig.3.20. We shall make the same model assumptions about the properties of tanks and pipes as in Chapter 1, with the exception that induction will be included. In our models, tanks only have the property of capacitance—we simply neglect that there is fluid friction as well, and that a tank might also be an inductor. The flow of the fluid through the pipe shows resistive and inductive behavior, as discussed in Section 3.2. We may represent our model in terms of an electric circuit including these same elements—namely a capacitor, an inductor, and a resistor (Fig.3.21).

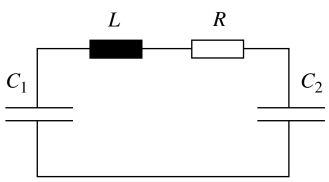


FIGURE 3.21. The electric circuit representing a model of the communicating tanks in Fig.3.20.

We already know all the important elements to create the system dynamics model of the oscillatory process. First, in Section 1.1, we created a model for communicating lakes (Fig.1.11). This model contains two storage elements representing the amounts of water stored in each of the lakes. Knowing the volume of water we can calculate the pressure at the bottom of the lakes from their capacitances. The difference of these pressure values is like the voltage of a battery in a circuit containing a resistor and an inductor (Fig.3.9). Knowing this value, we can calculate the current through these elements as in the model presented in Fig.3.13. The current is equated to the flux of fluid flowing from one tank to the other. Therefore, by simply combining the system dynamics diagrams in Figs.1.11 and 3.13, we obtain the desired model (Fig.3.22).

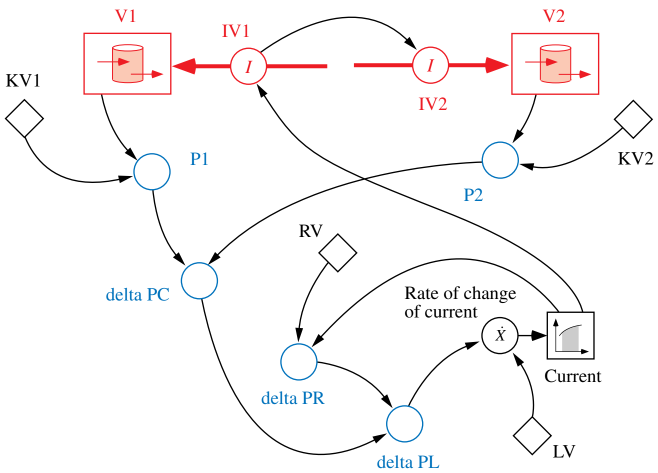


FIGURE 3.22. The combination of a model for communicating lakes neglecting induction, and the model representing an *LR*-circuit creates an oscillatory system.

Simulation of the model shown in Fig.3.22 leads to a solution just like the one in the graph of Fig.3.17 which demonstrates that we are on the right track. Simple oscillations occur in systems combining capacitors, inductors, and resistors (so-called *LCR*-systems). If the resistance R_V is set to zero in the model, the oscillation will be undamped, which is just what we should expect.

Description of Oscillatory Behavior

Let us now describe the behavior of the most simple oscillatory system in more detail. Experiments and models of *LC*-systems show that the oscillation—described by one of the important variables such as fluid volume in a tank or pressure, or charge, current or voltage in an electric circuit—takes the form of a sine-curve (Fig.3.23). This is often called a *harmonic oscillation*.

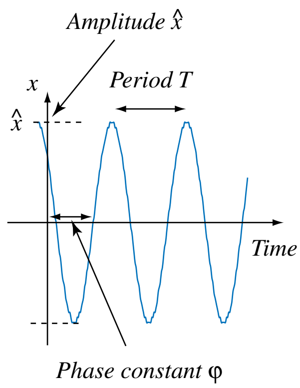


FIGURE 3.23. A simple harmonic oscillation is described in terms of a sine curve.

Descriptive quantities. There are three quantities which fix the function of time describing the solution of the oscillatory problem. The first is the *amplitude* of the oscillation which is half the magnitude of the difference between maximum and minimum of the sine-curve (Fig.3.23). The second is the *period* T of the oscillation, or its *frequency* f. Period and frequency are simply related by

$$T = \frac{1}{f} \tag{3.11}$$

Finally, the shift of the curve with respect to the origin is called the *phase constant* φ. With these three quantities, the oscillation can be represented by

$$x(t) = \hat{x}\sin(2\pi f t - \varphi) \tag{3.12}$$

Here, x stands for any of the quantities which change accordingly, such as the volume or level of fluid in a tank, the pressure at the bottom of a tank, the flux of fluid, or electric quantities such as charge, voltage, and current, and so forth. Often, the *angular frequency* ω is introduced instead of the frequency:

$$\omega = 2\pi f \tag{3.13}$$

With its help harmonic oscillatory behavior can be represented as follows:

$$x(t) = \hat{x}\sin(\omega t - \varphi) \tag{3.14}$$

Determining the period of oscillation. It is interesting to ask which system parameters the period or angular frequency of simple harmonic oscillations depend upon. We will be able to motivate the answer without setting up all the equations of an oscillatory system and solving them at this point. Refer to the example of two communicating tanks discussed above (Fig.3.20), and consider the fluid current as a function of time for the first quarter of a period. During this time interval, the current starts from zero to reach its highest value, and the levels of fluid in the tanks have become equal. Two phenomena combine to create this result. At the beginning, the inductive process of starting a current in an *LR*-system leads to the initial rise of the function. This happens on a time scale measured by the inductive time constant τ_L derived in Eq.(3.6). At the same time, the left container is discharging, and the right one is being filled, all of this on a time scale measured by the capacitive time constant τ_C (Chapter 2). Since the period of oscillation is expected to increase if the inductance, and therefore the inductive time constant, are increased, we can assume T to grow with τ_L. The same must be true for the relation between the period and the capacitive time constant. Increasing the capacitance of the system should lead to an increase of the period. In summary, we should expect the square of the period to be proportional to the product of both time constants:

$$T^2 \sim \tau_L \tau_C$$

Since $\tau_L = L_V/R_V$, and $\tau_C = R_V K_V$, the period of oscillation should be proportional to the square root of the product of L_V and K_V. This is indeed the case. As we shall see in Section 3.6, this is indeed the case. The result will be found to be

$$T = 2\pi\sqrt{L_V K_V} \tag{3.15}$$

Naturally, the results can be transferred to electric phenomena. We simply replace volume by charge, pressure by electric potential, hydraulic inductance by the electric counterpart, and so forth.

Q

1. Are the tanks or capacitors in Figs.3.20 and 3.21, respectively, in series or in parallel? What, therefore, is the total capacitance of the system of two containers?
2. A system composed of an electric capacitor and a solenoid is the simplest kind of an electromagnetic oscillator. Which quantity oscillates back and forth? Where are the two storage devices we said an oscillatory system should have? Which element shows inductive behavior?
3. What is the period of oscillation of mercury in the U-tube shown in Fig.3.17? What is the range of values of the period of oscillation of the El Niño phenomenon?